15-Sept

Chapters:

DISCRETE MATHEMATICS

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9 Marks

1. Logic
2. Combinatorics
3. Set Theory [KOLMAN, BUSAN $ ROSS]
4. Graph Theory [NARSINGH DEO] [Theory]

LOGIC

1. Logical Statement? [Proposition]
→ 2. Logical Operators & their properties (V, Λ, ¬, ⇒, ⇔, ⊕, ↑, ↓) — Disjunction, Conj., Negation, Implication, by Conditions, XOR, NAND, NOR
3. Tautology (T) (Valid), Contradiction (C) & Contingency (CT) [Satisfiable (T or CT) / Unsatisfiable (C)]
→ 4. Normal Forms & properties: PDNF (Principle Disjunctive Normal Form) — Sum of Min term
   PCNF (Principle Conjunction Normal Form) — Product of Max term.
Imp → 5. Implications $ Biconditional (⇒, ⇔)
6. Arguments & Fallacy [Invalid Argument]
7. Rules of Inference (9+4)
→ 8. Predicate Logic - Quantifier (∀, ∃)
   - Validity of a predicate
   - Properties
   - Translation — Eng → simbolic logic; S.L → English

LOGIC

(S, o)
↓ ↳ operators
Set of all Logical Stmt

Logical StateMent - (Preposition)

- Declarative sentence which can be either true or false but not both.

Ex - This board is white.
This Fan is Rotating.
- This sentence is true.

[is/will tends to declaration].

Not a Logical StateMent

1. Questions - What is Your Name?
2. Command - Stand up.
3. Exclamation - Oh! That's great.
4. $x+2=4$ Negative self Reference
   (it is not preposition bcoz for some x value it is true
   it is false)
5. He is tall. (unless he is specified)
6. Today is Wednesday. VAGUE
   [Not a preposition bcoz today may be true, but tommorow it will become false]
7. Tommorow it will rain.
   [Not a preposition.]
8. This sentence is false.
   [Negative Self Referential Sentence]

Logical Operators: (Logical Connectives) [$\vee, \wedge, \sim, \Rightarrow, \Leftrightarrow, \oplus, \uparrow, \downarrow$]

A preposition is written in the following way:

$p: 2+2=4$

$q(x): x+2=4$ (Predicate) but not a preposition

~~True~~ False $- \forall x\, P(x)$ } - prepositions
True $- \exists x\, P(x)$

Negation - ($\sim, \bar{p}, \neg, p'$) . Unary operators

P	$\bar{P}$
0	1
1	0

P	Negation
is	is not
is not	is
=	≠
<	≥
>	≤
$p\vee q$	$p'\wedge q' \Rightarrow p\downarrow q$
$p\wedge q$	$p'\vee q' \Rightarrow p\uparrow q$

$x>5,\ x\le 5$

* $\boxed{p \Rightarrow q \equiv p'+q}$

P	Negation
$p\Rightarrow q$	pq'
$p\Leftrightarrow q$	$p\oplus q$
$p\oplus q$	$p\Leftrightarrow q$
$p\uparrow q$	$p\wedge q$
$p\downarrow q$	$p\vee q$

$\boxed{p \uparrow q = (p.q)' = p'+q'}$

$\boxed{p \downarrow q = (p+q)' = p'q'}$

$\boxed{p \Leftrightarrow q \equiv pq + p'q'}$

$\boxed{p \oplus q \equiv pq' + p'q}$

$\boxed{p \oplus q = \sim(p \Leftrightarrow q)}$

$p' = p \uparrow q$

$p + q = (p \uparrow p) \uparrow (q \uparrow q)$

~~p.q (q)~~

- if $p \vee q = 1$

  than $p = \neg q$ is one possibility but not the sure thing. it also allow some other thing

- if $p \wedge q = 0$

  $\Rightarrow [p = \neg q]$ not allways.

- If $p \vee q = 1$ & $p \wedge q = 0 \Rightarrow \boxed{p = \neg q}$

- If $p \Rightarrow 2+2=4$ or $3+7=10$

  $\bar{p} \Rightarrow 2+2 \neq 4$ and $3+7 \neq 10$

- If $p \Rightarrow 2+2=4$ and $3+7=10$

  $\bar{p} \Rightarrow 2+2 \neq 4$ OR $3+7 \neq 10$

$\wedge$ - all must be true

$\vee$ - atleast one is true

$\oplus$ - Exactly one is true

$\uparrow$ - Atleast one is false

- p: 2 is even & divisible by 4.

  p': 2 is odd or not divisible by 4.

- p: if it rains, i will carry umbrella. [Either it does not rain OR I will carry Umbrella]

  p': It rains and I will not carry Umbrella

Conversion of Secondary operators into Basic operators:

- $p \rightarrow q = p' + q$
- $p \Leftrightarrow q = p'q' + pq = (p \oplus q)' = p' \Leftrightarrow q' = p' \oplus q = p \oplus q'$
- $p \oplus q = pq' + qp' = p' \Leftrightarrow q = p \Leftrightarrow q' = p' \oplus q'$
- $p \Leftrightarrow q = (p'+q)(p+q')$ $[(p \Rightarrow q) \wedge (q \Rightarrow p)]$
- $p \oplus q = p \oplus q'$
- $p' \oplus q = p \Leftrightarrow q = p \oplus q'$

- p: A number is even if and only if divisible by 2. $[p \Rightarrow q \wedge p \Leftarrow q]$

  P': A number is even or it is divisible by 2, but not both.

- NOR - Neither ... NOR
- OR - Either .... OR

Negation for predicate-

$P(x)$	$\neg P(x)$
$\forall x P(x)$	$\exists x \neg P(x)$
$\exists x P(x)$	$\forall x \neg P(x)$
$\forall x \neg P(x)$	$\exists x P(x)$
$\exists x (\neg P(x))$	$\forall x P(x)$

$\sim(\forall x(P(x) \to Q(x))) \equiv \exists x(\sim[P(x) \to Q(x)])$

$\equiv \exists x(P(x) \wedge \sim Q(x))$

$\sim(\forall x \exists y\ P(x,y)) \equiv \exists x \forall y \sim P(x,y)$

$\sim(\exists x \forall y \forall z\ (P(x,y,z) \oplus Q(x,y,z))) \equiv \forall x \exists y \exists y\ (P(x,y,z) \Leftrightarrow Q(x,y,z))$

$\sim(p \Rightarrow q) = \sim(p'+q)$

$p \Rightarrow q$	[stmt].	$(p=0 \text{ or } q=0) \Rightarrow (pq=0)$
$q \Rightarrow p$	[converse]	$(pq \neq 0) \Rightarrow (p=0 \text{ or } q=0)$
$\sim p \Rightarrow \sim q$	[inverse]	$(p \neq 0 \text{ and } q \neq 0) \Rightarrow (pq \neq 0)$
$\sim q \Rightarrow \sim p$.	[contrapositive]	$(pq \neq 0) \Rightarrow (p \neq 0 \text{ and } q \neq 0)$

binary operators-

p	q	$p \vee q$ ($p+q$)	$p \wedge q$ ($p \cdot q$)	$p \Rightarrow q$	$p \Leftrightarrow q$	$p \oplus q$	$p \uparrow q$	$p \downarrow q$
0	0	0	0	1	1	0	1	1
0	1	1	0	1	0	1	1	0
1	0	1	0	0	0	1	1	0
1	1	1	1	1	1	0	0	0

if two prepositions are equivalent (x, y)

then $[x \Leftrightarrow y \equiv 1]$ $[x \equiv y \text{ iff } x \Leftrightarrow y = 1]$

Q ~~let~~ $b \Leftrightarrow c$ and $a \Leftrightarrow (b \vee \neg b)$ is tautology

what can be inferred about $a \vee (b \wedge c)$?

$b \Leftrightarrow c \Rightarrow b \equiv c$

$a \Leftrightarrow (b \vee \sim b) \Rightarrow a = 1$

$\therefore a \vee (b \wedge c) = a \vee (b \wedge b) = 1 \vee b = 1$ (Tautology)

boolean Algebra: $(S, +, \cdot, ')$ [logic, Digital logic, Set theory]

$(S, \vee, \wedge, \sim)$

$(S, \cup, \cap, A^c)$

- No of elements in set of Boolean Algebra must be in power of 2.
- Q_n is a boolean Algebra.

- $a - b = a \wedge b'$
- $A - (B \cup C) = (A-B) \cup (A-C)$. (Test T or F)

$a - (b+c) = (a-b) + (a-c)$

$a b'c' = ab' + ac'$ (false)

Properties of Operators-

• Operators are also known as Logical connectives.

1. **Closure-** $\forall x, y,$ $\begin{bmatrix} x+y \in S \\ x \cdot y \in S \end{bmatrix}$ $\neg x \in S$ OR $\begin{bmatrix} x \vee y \in S \\ x \wedge y \in S \end{bmatrix}$ $\neg p \in S$

$\forall A, B \in S$ $\begin{bmatrix} A \cup B \in S \\ A \cap B \in S \\ A^c \in S \end{bmatrix}$

2. **Commutative:**

$\forall x, y \in S$ $\begin{bmatrix} x+y = y+x \\ x \cdot y = y \cdot x \end{bmatrix}$ $\forall A, B \in S$ $\begin{bmatrix} (A \cup B) = (B \cup A) \\ (B \cap A) = (A \cap B) \end{bmatrix}$

$\forall x, y \in S$ $\begin{bmatrix} x \wedge y = y \wedge x \\ x \vee y = y \vee x \end{bmatrix}$

3. **Associative:**

$\forall x, y, z \in S$ $\begin{bmatrix} x+(y+z) = (x+y)+z \\ x \cdot (y \cdot z) = (x \cdot y) \cdot z \end{bmatrix}$

$\forall x, y, z \in S$ $\begin{bmatrix} x \wedge (y \wedge z) = (x \wedge y) \wedge z \\ x \vee (y \vee z) = (x \vee y) \vee z \end{bmatrix}$

$\forall A, B, C \in S$ $\begin{bmatrix} A \cup (B \cup C) = (A \cup B) \cup C \\ (A \cap B) \cap C = A \cap (B \cap C) \end{bmatrix}$

4. **Distributive:** $\forall x, y, z \in S$ $\begin{bmatrix} x+(y \cdot z) = (x+y)(x+z) \\ x \cdot (y+z) = xy + xz \end{bmatrix}$

$\forall x, y, z \in S$ $\begin{bmatrix} x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \end{bmatrix}$

$\forall$ ~~x, y, z~~ A, B, C $\in S$ $\begin{bmatrix} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{bmatrix}$

5. **Identity:** $\begin{bmatrix} \exists_0 \forall x \quad x+0 = x = 0+x \\ \exists_1 \forall x \quad x \cdot 1 = x = 1 \cdot x \end{bmatrix}$ & $0 \neq 1$

$\forall x \in S$ $\begin{bmatrix} x \wedge T = x = T \wedge x \\ x \vee F = x = F \vee x \end{bmatrix}$

$\begin{matrix} \exists_\phi \; \forall A \in S \\ \exists_S \; \forall A \in S \end{matrix}$ $\begin{bmatrix} A \cup \phi = A = \phi \cup A \\ A \cap S = A = S \cap A \end{bmatrix}$ [S → Universal set]

Ques- The Smallest Finite Boolean Algebra has 2^n. what is value of n?

$n = 1 \Rightarrow 2^n \{0, 1\}$ → Greatest Elmt, ↳ Least Element

[+ → Least Upper Bound; · → Greatest Lower Bound]

Complement -

$\forall x \exists x' \begin{bmatrix} x + \bar{x} = 1 \\ x \cdot x' = 0 \end{bmatrix}$

$\begin{bmatrix} x \wedge (\sim x) = F \\ x \vee (\sim x) = T \end{bmatrix} \forall x \exists \sim x$

$\forall A \in S \; \exists A^c \in S \begin{bmatrix} A \cup A^c = S \\ A \cap A^c = \phi \end{bmatrix}$ [S - Universal set]

Precedence of operators:

$[() > (') > \wedge > \vee]$ → for boolean Algebra

$() > ' > \cdot > + > \Rightarrow > \Leftrightarrow$

Ex- $(\,[\,(p \vee ((\sim q) \wedge r)) \Rightarrow s\,] \Leftrightarrow t)$

If an element satisfy complement property it surely satisfies identity property.

Laws for Boolean Algebra-

Idempotent Law - $\forall p \in S \begin{bmatrix} p + p = p \\ p \cdot p = p \end{bmatrix}$ $\forall p \in S \begin{bmatrix} p \wedge p = p \\ p \vee p = p \end{bmatrix}$

$\forall A \in S \begin{bmatrix} A \cup A = A \\ A \cap A = A \end{bmatrix}$

- The biggest polynomial in Boolean Algebra with degree 1.
- No power, No coefficient exist

Absorbtion Law

$\forall p \in S \begin{bmatrix} p + pq = p \\ p \cdot (p + q) = p \end{bmatrix}$ $\begin{bmatrix} p + qp = p = p + pq \\ p \cdot (p + q) = p = (p + q) \cdot p \end{bmatrix}$

Ex - $pq + pqr + pr\bar{s}t + p$
$= pq + pqst + p$
$= pq + p = p$

Ex4 $p + [(p + q')(q' + s')]$
$\neq p$

Ex $p \cdot (q + p'r)$
$\neq p$

Ex3 $[p \cdot (q + r's' + p + t' + q'r') = p]$

$\forall p \in S$ $\begin{bmatrix} p+p'q = p+q \\ p(p'+q) = pq \end{bmatrix}$ $\forall p \in S$ $\begin{bmatrix} p'+pq = p'+q \\ p'(p+q) = p'q \end{bmatrix}$

Ex1 - $p+q'r+r+p'q$

$= r+p+q$

3) Demorgan's Law : $\forall p,q \in S$ $\begin{bmatrix} (p+q)' = p'q' \\ (p\cdot q)' = p'+q' \end{bmatrix}$ $\begin{bmatrix} \sim(p \vee q) = \sim p \wedge \sim q \\ \sim(p \wedge q) = \sim p \vee \sim q \end{bmatrix}$

Ex- $(p+qr) \Rightarrow s$

Simplified form is $[p+qr]'+s = p'\cdot(q'+r')+s$

for Set Theory- $\forall A,B \in S$ $\begin{bmatrix} (A \cup B)^c = A^c \cap B^c \\ (A \cap B)^c = A^c \cup B^c \end{bmatrix}$

4. Law of Double Complement :

$\forall p \in S$ $(p')' = p$

$\forall A \in S$ $(A^c)^c = A$

- $p' = q$ if and only if $q' = p$

$[p' = q \Leftrightarrow q' = p]$ Always a tautology

5. Domination Law- $\forall x \in S$ $\begin{bmatrix} x+1 = 1 \\ x\cdot 0 = 0 \end{bmatrix}$

$\forall p \in S$ $\begin{bmatrix} x \vee 1 = 1 \\ x \wedge 0 = 0 \end{bmatrix}$

$\forall A \in S$ $\begin{bmatrix} A \cup U = U \\ A \cap \phi = \phi \end{bmatrix}$ U- Universal set

Note- 1. $\underbrace{[p' \Rightarrow q] \equiv [q' \Rightarrow p]}_{\text{Contrapositive set}} \equiv (p \vee q)$

Tautology, Contradiction & Consistency

Ques: Which of the following is tautology?

(a) ~~$p \Rightarrow q$~~ $p \vee q \Rightarrow p \wedge q$ $\qquad$ (a) $(p+q) \Rightarrow pq = p'q' + pq = p \Leftrightarrow q$

(b) $p \vee q \Rightarrow r \wedge p$ $\qquad$ (b) $(p+q) \Rightarrow rp = p'q' + pr$

(c) $p \vee q \Rightarrow r \vee s$ $\qquad$ (c) $(p+q) \Rightarrow r+s = p'q' + r + s$

(d) $s \Rightarrow s \wedge t$ $\qquad$ (d) $s' + st = s' + t$

✓(e) None

Tautology - for a preposition, if truth table contain all true values only
OR
$[p \Leftrightarrow 1 \equiv 1]$ • Also known as valid preposition.

Ques - Check whether it is a tautology?

$$[[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (r \Rightarrow p)]$$

$$[[(p'+q)(q'+r)] \Rightarrow (r'+p)]$$

$$= (p'+q)' + (q'+r)' + r' + p$$

$$= pq' + qr' + r' + p$$

$$= p + r' \quad \Rightarrow \text{Hence it is contigency}$$

Ques - Check whether it is tautology?

$$[((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)]$$

$$= pq' + qr' + p' + r$$

$$= p' + q' + r + q = 1 \quad \text{Hence it is a tautology}$$

Contradiction - for all value, it is false $\quad [p \Leftrightarrow 0 \equiv 0]$

- Also known as fallacy.
- Also known as invalid.

[Preposition]
- [Satisfiable (SAT)]
  - [Tautology] (T) [All true]
  - [contingency] (CT) $\neq 0$ or 1
- [Unsatisfiable (USAT)]
  - [Contradiction] (C) $= 0$

Ques- Which of the following is valid?

1. $p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$
2. $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

1.> $p' + q + r \equiv (p'+q)+(p'+r)$

$\equiv p'+q+r$ (valid)

2. $p' + qr \equiv (p'+q)(p'+r)$

$\equiv p'+qr$ (valid)

Ques- Check whether valid or not?

$A \cup (B - C) \equiv (A \cup B) - (A \cup C)$

$A + BC' \equiv (A+B)(A'C')$

$A + BC' \equiv A'BC'$

↳ Hence it is false / invalid

Note- 1. Both Commutative & Associative

(AND, OR, XOR, $\Leftrightarrow$)

Commutative: $A \Leftrightarrow B \equiv B \Leftrightarrow A$

$A'B' + BA \equiv AB + A'B'$

Associative: $A \Leftrightarrow (B \Leftrightarrow C) \equiv (A \Leftrightarrow B) \Leftrightarrow C$

$A \Leftrightarrow (BC + B'C') \equiv (AB + A'B') \Leftrightarrow C$

$ABC + AB'C' + A'(B'+C')(B+C) \equiv ABC + A'B'C + (A'+B')(A+B)C'$

$ABC + AB'C' + A'B'C + A'BC' \equiv ABC + AB'C' + A'B'C + A'BC'$

Ques- Check whether it is valid or not?

$(p \Leftrightarrow q) \wedge (p' \Leftrightarrow q)$

$\Rightarrow$ if $z = p \Leftrightarrow q$ $\quad \Rightarrow z \wedge z' = 0$ Hence it is invalid.

$z' = p' \Leftrightarrow q$

2. Neither Commutative NOR Associative (implication ($\Rightarrow$))

- $p \Rightarrow q \neq q \Rightarrow p$
- $p \Rightarrow (q \Rightarrow r) \neq (p \Rightarrow q) \Rightarrow r$

Commutative- $p' + q \neq q' + p$

Associative: $p' + q' + r \neq pq' + r$

3) Commutative but not Associative [NAND(↑), NOR(↓)]

Commutative: $p \uparrow q \equiv q \uparrow p \Rightarrow p' + q' = q' + p'$

$p \downarrow q \equiv q \downarrow p \Rightarrow p'q' = q'p'$

Associative: $\begin{bmatrix} p \uparrow (q \uparrow r) \not\equiv (p \uparrow q) \uparrow r \\ p' + qr \neq pq + r' \end{bmatrix}$

$\begin{bmatrix} p \downarrow (q \downarrow r) \not\equiv (p \downarrow q) \downarrow r \\ p'(q+r) \neq (p+q)r' \end{bmatrix}$

Functionally Complete Set - Set of operators which is capable to write ~~Any~~ every boolean function using it.

Ex - $\{ \vee, \wedge, \sim \} \rightarrow$ FC Set

$\{ \vee, \wedge, \sim, \Rightarrow, \Leftrightarrow, \uparrow, \downarrow \} \rightarrow$ FC Set

$\{ \vee, \wedge \} \rightarrow$ Not a functionally Complete Set

Minimal FC Set
- $\{ \vee, \sim \} \rightarrow$ FC Set
- $\{ \wedge, \sim \} \rightarrow$ FC Set
- $\{ \Rightarrow, \sim \} \rightarrow$ FC Set as $\begin{bmatrix} p \Rightarrow q = p' + q \\ p' \Rightarrow q = p + q \end{bmatrix}$

$\{ \Leftrightarrow, \sim \} \rightarrow$ Not FC Set $\quad p \Leftrightarrow q = p'q' + pq$

$\{ \uparrow \} \rightarrow$ FC Set
$\{ \downarrow \} \rightarrow$ FC Set
[Smallest Minimal FC Set]

$\{ \uparrow \}$ -
- $\sim \Rightarrow (p \uparrow p) = \sim p$
- $\vee \Rightarrow (p \uparrow p) \uparrow (q \uparrow q) = p \vee q$
- $\wedge \Rightarrow (p \uparrow q) \uparrow (p \uparrow q) = p \wedge q$

$\{ \downarrow \}$ -
- $\sim \Rightarrow (p \downarrow p) = \sim p$
- $\wedge \Rightarrow (p \downarrow p) \downarrow (q \downarrow q) = p \wedge q$
- $\vee \Rightarrow (p \downarrow q) \downarrow (p \downarrow q) = p \vee q$

Minimal FC Set -
- it should be FC
- No Subset of FC should be FC.

Ex - 1. $\{ \vee, \wedge, \sim \} = FC$

$FC_1 \subset FC = \{ \vee, \sim \}$ also an FC

$\therefore$ it is not minimal

Smallest MFC - Minimal functionally complete set with smallest cardinality

MFC Set = $\{\uparrow\}$ $\{\downarrow\}$

Tautology (T), Contingency (CT), Contradiction (C)

A well formed formula can be T, C or CT (collectively exhaustive & mutually exclusive)

- wff ≡ Logical stmt ≡ logical f^n ≡ boolean function ≡ logical Exprsn.

A well formed formula can be either ~~con~~ Satisfiable (SAT) OR unsatisfiable (USAT).

Note-
1. Contigency ⇒ Satisfiable
2. Tautology ⇒ Satisfiable
3. Every Satisfiable is a contigency
   Every Satisfiable is a Tautology } False
4. Every Contradiction is Unsatisfiable

   Contradiction (Every) ⇔ Unsatisfiable (Every)

5. The Complement of tautology is always a contradiction (USAT)
6. ________ Contigency is always a contingency (SAT)
7. ________ Contradiction is always a tautology. (SAT)

Contingency- atleast one row should be 0 & atleast one row should be 1, in truth table.

Ques- if the truth table for p contains atleast 1 row is zero, what can we say about it?
It can be either contingency or contradiction but not tautology.

8. If ~~a~~ p is unsatisfiable, then $\neg p$ will be satisfiable & ~~cont~~ tautology.
9. If p is satisfiable, p' may be satisfiable (p is ~~taut~~ contigency) or Unsatisfiable (P is tautology)

Ques- p is satisfiable, $\neg p$ is also satisfiable
what is p?

(a) C
(b) T
(c) CT ✓

Note- 1. p is SAT, ~p is SAT ⟺ p is (CT)

2. p is SAT, ~p is USAT ⟺ p is (F)

NORMAL FORMS: Writing of the std. exprsn. with the boolean operators for a given exprsn.

- **PDNF** -
  - Principal Disjunctive Normal Form
  - Min term present in exprsn
  - V distributed over ∧.
  - min terms are denoted by m_i

$\begin{bmatrix} 0 \to p' \\ 1 \to p \end{bmatrix}$

Ex- $p \Leftrightarrow (q \Rightarrow r')$

pdnf: $p'qr + pq'r' + pq'r + pqr'$

pcnf: $(p+q+r)(p+q+r')(p+q'+r)$
$(p'+q'+r')$

p	q	r	$(q \Rightarrow r')$	$p \Leftrightarrow (q \Rightarrow r')$
0	0	0	1	0 - M_0
0	0	1	1	0 - M_1
0	1	0	1	0 - M_2
0	1	1	0	1 - m_3
1	0	0	1	1 - m_4
1	0	1	1	1 - m_5
1	1	0	1	1 - m_6
1	1	1	0	0 - M_7

PCNF - Principal Conjunctive NorMal Form
- Max term present in exprsn
- ∧ distributed over V
- denoted by M_i
- 0 is in uncompleted form & 1 is in complemented form.

Note- 1. If either of PDNF or PCNF form contains zero terms than expr is either satisfiable or Unsatisfable

Zero terms in PDNF ⇒ Unsatisfiable [Contradiction]

Zero terms in PCNF ⇒ Satisfiable [Tautology]

Non Zero PCNF & PDNF ⇒ Contingency [Satisfiable]

Properties:
1. PDNF & PCNF for an exprsⁿ will be unique.
2. PDNF & PCNF provides ease for circuitory Design.
3. To prove $x = y$, we can say PDNF(X) = PDNF(Y)
OR
PCNF(X) = PCNF(Y)

$$\boxed{X \equiv Y \Leftrightarrow PDNF(X) = PDNF(Y) \text{ or } PCNF(X) = PCNF(Y)}$$

4. No of terms in PDNF (minterm) + No of terms (maxterm) in PCNF $= 2^n$

Ex- for a 5 variable expⁿ if no of terms in PDNF (x) = 10 what is the no of term in PCNF(X)?

$\boxed{PCNF(X) = 2^5 - 10 = 22}$

5. Max No of possible Minterm in n-variable function $= 2^n$ when $f(x_1 \dots x_n) = 1$
6. Max possible max-terms in n-variable function $= 2^n$ when $f(x_1 \dots x_n) = 0$
7. No of distinct possible PDNF for n variable $= 2^{2^n}$
No of distinct possible PCNF for n variable $= 2^{2^n}$
8. No of Boolean function possible for n Variable $= 2^{2^n}$
No of Truth table possible for n variable $= 2^{2^n}$

Ques for given TT what can you say about ↺
I. commutative
II. Associative

p	q	p↺q
0	0	0
0	1	0
1	0	1
1	1	0

[pq']

1. Commutative - $p \circlearrowleft q = q \circlearrowleft p$
$pq' = qp'$ (false)

2. Associative - $p \circlearrowleft (q \circlearrowleft r) = (p \circlearrowleft q) \circlearrowleft r$
$p \circlearrowleft (qr') = (pq') \circlearrowleft r$
$\Rightarrow p(q' + r) = pq'r'$ (false)

9) No of tautology for n variable $f^n = 1$
No of contradiction for n variable $f^n = 1$
No of Contingency for n variable $f^n = 2^{2^n} - 2$

9. probability for a f^n to be tautology for n variable $= 1/2^{2^n}$

——— contradiction for n variable $= 1/2^{2^n}$

——— contingency for n variable $= 2^{2^n} - 2/2^{2^n}$

$= [1 - 2^{1-2^n}]$

10) probability that a f^n of n variable is satisfiable $= 1 - 2^{-2^n}$

11) probability that ——— is unsatisfiable $= 2^{-2^n}$

12) No of ~~Min~~ terms in tautology PDNF $= 2^n$
PCNF $= 0$

No of ~~Max~~ terms in contradiction PDNF $= 0$
PCNF $= 2^n$

No of terms in contingency PDNF(minterm) $= 1 \leq x \leq 2^n - 1$
PCNF (max term) $= 1 \leq x \leq 2^n - 1$

13) for a given assignment of truth value for p, q, r ($p=1, q=0, r=1$)

the No of min terms/clauses which is equal to 1 $= 1$

the No of min terms/clauses which ~~equo~~ evaluates to 0 $= 7 (2^n - 1)$

No of Max terms evaluates to 0 $= 1$

——— evaluates to 1 $= 7 (2^n - 1)$

Ques- what fraction of Pcnf ~~for equio~~ [clauses] evaluates to 1?

(a) atleast ½ — for n variable, pcnf clauses evaluates to 1 $= \frac{2^n - 1}{2^n}$

(b) atmost ½

(c) atleast ¼

(d) atmost ¼

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv p \wedge q \to r$

***

$[x \Rightarrow [y \Rightarrow z]] \equiv [x \wedge y \Rightarrow z]$

$x' + (y \Rightarrow z) \equiv (xy)' + z$

$x' + y' + z \equiv x' + y' + z$

$x \Rightarrow (y \Rightarrow z) \neq (x \Rightarrow y) \Rightarrow z$

Implication & Biconditional

$p \Rightarrow q$

1. p implies q
2. If (p) than (q)
   - (p) → antecedant (Action)
   - (q) → consequent (Result)
3. q, If p | q follows from p
   from p, q follows
4. If p, q
5. $\begin{bmatrix} X \text{ if } Y = Y \to X \\ X \text{ only if } Y = X \to Y \end{bmatrix}$

- I will carry Umbrella only if it rains.
  $U \to R$
- I will carry Umbrella, if it rains
  $R \to U$
- You can enter the theatre only if you have the ticket.
  Theatre ⇒ ticket

6. q follows from p
   Ex- $pq=0$ follows from $p=0, q=0$
   $[p=0, q=0] \Rightarrow pq=0$
7. from p, q follows.
   Ex- Only if it rains, I will carry Umbrella.
   I will carry umbrella, only if it rains
   ⇓
   [Umbrella ⇒ rains]
8. q is neccessary for p.
9. p is sufficient for q.
10. for p, q is neccessary.
    It is neccessary to eat for health
    health ⇒ eat
11. For q, p is sufficient.
12. $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$
13. $q \Rightarrow p \equiv \neg p \Rightarrow \neg p$

$p \Leftrightarrow q$

1. if & only if
2. p if and only if q
3. p iff q
4. (p if q) and (q if p)
   $(p \Rightarrow q) \wedge (q \Rightarrow p)$
5. (p if q) and (p only if q)
   $(q \Rightarrow p) \wedge (p \Rightarrow q)$
6. p is neccessary & sufficient for q
7. $p \Leftrightarrow q \equiv \neg q \Leftrightarrow \neg p \equiv p \vee \neg p \Leftrightarrow \neg q$
   $\equiv q \Leftrightarrow p$
8. [Inverse ≡ converse ≡ Dired ≡ Contra positive]
9. $\left(\text{p is necessary for q}\right) \wedge \left(\text{p is sufficient for q}\right)$
   $(q \Rightarrow p) \wedge (p \Rightarrow q)$
10. p is same as q.
    Ex- Working hard is same as being rich
    [Work hard ⇔ Rich]
11. p is equivalent to q
    $[p \Leftrightarrow q] \equiv 1$

Some Translation :

why unless not resembling with if

1.) when $\equiv$ whenever $\equiv$ if

2.) But $\equiv$ and $\equiv$ Nevertheless

The sun is shining but the grass is still wet

p: sun is shining.

q: grass is still wet. $\therefore (p \wedge q)$

3.) Unless $\equiv$ OR

1. You will not pass the exam unless you study hard.

You will not pass the xam OR You work hard

p: You will pass the xam

q: You will study hard

$(\neg p \vee q)$

2.) I didnot study for xam nevertheless I passed.

$\underbrace{\text{I didnot study for}}_{p}$ xam nevertheless $\underbrace{\text{I passed}}_{q}$ $[p \wedge q]$

4.) as well $\equiv$ AND $\equiv$ also

1. I passed as well he passed.

(I passed) $\wedge$ (he passed)

2. I passed also he passed.

(I passed) $\wedge$ (he passed)

$\therefore$ p or q but not both $\equiv p \oplus q \equiv$ Exactly one of p and q

— Either p or q $\equiv p \vee q \equiv$ atleast one of p or q.

Neither p nor q $\equiv$ NOR $\equiv \neg p \wedge \neg q \equiv$ not p & not q

Either p is false or q is false $\equiv$ NAND $\equiv (\neg p \vee \neg q) \equiv$ atleast one of p or q is false

• atleast 1 is true	$p \vee q$
• atleast 1 is false	$p \uparrow q$.
• Exactly 1 is true	$p \oplus q$
• both are true	$p \wedge q$
• both are false	$p \downarrow q$

9. Required $\equiv$ neccessary $\equiv$ must

10. Enough $\equiv$ Sufficient

Degree is required for job.

Job $\Rightarrow$ Degree

Degree is sufficient for job [Degree $\Rightarrow$ Job]

1. You cannot ride on roller coaster if you are under 3 feet unless you are more than 10 yrs old.

p: You cant ride on roller coaster

q: Under 3 feet

r: more than 10 years

$[(q \Rightarrow \sim p) \vee r]$

[In English $\Rightarrow$ has more precedence than $\vee$]

2. P is neccessary but not suffiend for q.

$\underbrace{\text{Having Ticket}}_{p}$ is neccessary but not sufficient for $\underbrace{\text{entering to theatre}}_{q}$

(p is neccessary for q) but (p is not sufficient for q)

$(q \Rightarrow p) \quad \wedge \quad \sim(p \Rightarrow q)$

$= [(q \Rightarrow p) \wedge \sim(p \Rightarrow q)] = (q' + p)(pq') = pq' = p \wedge \sim q$

Note- i. DIRECT : $p \Rightarrow q$

CONVERSE – : $q \Rightarrow p$

CONTRAPOSITIVE : $\sim q \Rightarrow \sim p$

INVERSE : $\sim p \Rightarrow \sim q$

Ques- 'I will stay only if you go'

What is the converse for above stmt?

(a) I dont stay follows from you dont go. $[\sim q \rightarrow \sim p]$

(b) I stay is neccessary for you dont go. $[\sim q \rightarrow p]$

(c) I stay is sufficient for you go. $[p \Rightarrow q]$

(d) I stay follows from you go $[q \Rightarrow p]$

2) If $p \Rightarrow q$ is true/valid/holds

- i) if p is false, q can be either 0, 1.
- (ii) if p is true, q ~~can~~ will surely be true.
- (iii) if q is true, p can be either true or false
- (iv) if q is false, p is also false.

3. if p is False, $p \Rightarrow q$ is also true
If q is true, $p \Rightarrow q$ is always true.

Ques- if $p \vee q \Rightarrow r$ holds & r is false what can you say about p, q?

A- $p \vee q \Rightarrow 0 \Rightarrow p \vee q = 0$ ie p & q both must be false

Ques- if $p \vee q \Rightarrow r \Rightarrow t \wedge q$ holds & $p \vee q$ is true
[r, t, q All are true]

Ques- if $p \Rightarrow q$ is true
what is $\sim p \Rightarrow q \vee r$

$\sim p \Rightarrow q \vee r \Rightarrow p + q + r$

p	q	p+q+r
0	0	r
0	1	1
1	1	1

(a) T
(b) CT ✓
(c) C
(d) None

Ques If $p \Rightarrow q$ is false
what is $\sim p \Rightarrow q \vee r$?

$p \Rightarrow q = 0$
$\hookrightarrow p = 1, q = 0$

$\Downarrow$
$p + q + r$
$1 + 0 + r = 1$

(a) T ✓
(b) CT
(c) C
(d) None

Ques- If $[p \vee q \Rightarrow r \vee s \Rightarrow s \vee t] \Rightarrow (p \vee \sim p)$
what is the expression equivalent to?
(a) T ✓
(b) CT
(c) C
(d) None

Note- For any argument, if [one is premise is false] OR [Conclusion is true] $\Rightarrow$ always a valid argument.

2. To check the validity of an argument -
– Assume LHS to true
& then check ~~if~~ q is true $\Rightarrow$ ~~true~~
else false

Ques- check for the validity

$$\forall x (P(x) \land Q(x)) \Rightarrow (\forall x (P(x)) \land (\forall x Q(x))$$

Let LHS is true ie $P_1 \land Q_1 = T$
$P_2 \land Q_2 = T$
$\vdots$
$P_n \land Q_n = T$

RHS- $\forall x\ P(x)$ is true as all $P_1 \dots P_n$ is true
$\forall x\ Q(x)$ is true as $Q_1, Q_2 \dots Q_n$ is true

$\therefore$ RHS is true

Hence it is valid.

Ques- check for the validity

$$\forall x (P(x) \lor Q(x)) \Rightarrow [\forall x\ P(x)] \lor [\forall x\ Q(x)]$$

L.H.S : Let LHS is true
$P_1 \lor Q_1$ is true Let $P_1 = T$, $Q_1 = F$
$P_2 \lor Q_2$ is true Let $P_2 = F$, $Q_2 = F$

$\therefore$ It is invalid

R.H.S $\forall x\ P(x) \Rightarrow$ false for P_2
$\forall x\ Q(x) \Rightarrow$ false for Q_1
$\Rightarrow$ Hence false

Arguments : A set of Premises followed by a conclusion.

$$[(P_1 \land P_2 \land P_3 \dots P_n) \vdash Q]$$

P_1
P_2
P_3
$\vdots$
$\therefore Q$

If $P_1 \dots P_n$ is true than Q must be true.

An Argument is valid if conjunction of all premises implies q

$$[((P_1 \land P_2 \land P_3 \land \dots P_n) \Rightarrow Q) \equiv 1]$$

Argument- 1. If you work hard, you will pass
2. You are working hard
3. therefore, You will pass the xam.

$p \Rightarrow q$
p
$\therefore q$

[it is valid]

Ques 1. If you work hard, You will pass
2. You pass
3. therefore, You work hard

$p \Rightarrow q$
q
$\therefore q \neq p$

[It is invalid]

ues - Check whether it is valid or not:

If $\underbrace{\text{you work hard}}_{p}$ & $\underbrace{\text{you are talented}}_{q}$ then $\underbrace{\text{you will become musician}}_{r}$

If you become musician, then $\underbrace{\text{you will be happy}}_{s}$

You work hard

You are not talented

therefore I will be happy.

$p \wedge q \Rightarrow r$
$r \Rightarrow s$
p
$\sim q$
$\sim s$

$p \wedge q \Rightarrow r$
$r \Rightarrow s$
$p \wedge q \Rightarrow s$

$p \wedge q \Rightarrow s$
p
$\sim q$
$s.1$ Hence invalid

[because you cant make a comment about s]

You are talented

2) therefore You are happy.

$p \wedge q \Rightarrow r$
$r \Rightarrow s$
p
q
s

$\Rightarrow p \wedge q \Rightarrow s$
p, q
s

$\Rightarrow$ true

3) $p \vee q \Rightarrow r$
$r \Rightarrow s$
$\sim s$
$\sim p$

$p \vee q \Rightarrow s$
$\sim s$
$\sim p$ → valid as if $\sim s$ $p \vee q \Rightarrow F$
$\therefore \sim p$

4) $p \vee q \Rightarrow r$
$r \Rightarrow s$
s
p invalid (cant make a sure comment)

Ques - If $\overbrace{\text{you went to party}}^{p}$, You will have $\overbrace{\text{soup}}^{q}$ or $\overbrace{\text{salad}}^{r}$

You did not have soup

therefore, You will have salad.

$p \Rightarrow (q \vee r)$
$\sim q$
r
Not sure.

Ques- If you went to party, you will have soup or salad
You went to party (without this ① is invalid with this valid.
You did not have soup

①

$P \Rightarrow S \cup SA$
P
SA
$\sim S$ (Invalid)

$P \Rightarrow S \oplus SA$
P
SA
$\sim S$ (Valid)

$P \Rightarrow S \otimes SA$
P
SA
S (Invalid)

$P \Rightarrow S \oplus SA$
P
$\sim SA$
S (Valid)

$P \Rightarrow S \cup SA$
P
$\sim SA$
S Valid

$P \Rightarrow S \downarrow SA$
P
$\sim S$

$P \Rightarrow S \downarrow SA$
P
$\sim SA$
(neither soup or salad)

$P \Rightarrow S \uparrow SA$
P
$\sim S$ [invalid]

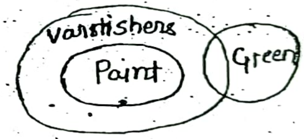
$P \Rightarrow S \vee SA$
P
$\therefore S \wedge SA$ [invalid]

$P \Rightarrow S \wedge SA$
P
$S \vee SA$ [Valid]

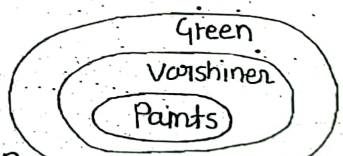
$$\boxed{p \wedge q \rightarrow p \rightarrow p \vee q}$$

Ques Predicate Arguments:

1. 1. All ~~parents are~~ paints are varnishers
2. Some varnisher are green
3. therefore Some patents are green [False]



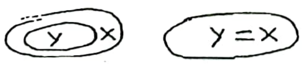
2. All paints are varnisher
All varnishers are green
therefore some varnishers are green



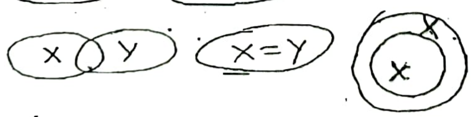
[Valid]

3. All X is Y

4. All Y is X

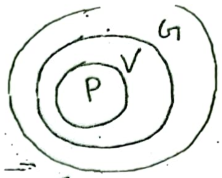


5. Some X is Y



6 Some Y is X.

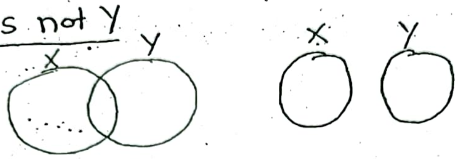
Ques- 1. All paint are varnisher
2. All varnisher is green
3. ∴ Some paint is not green



$P = V = G$

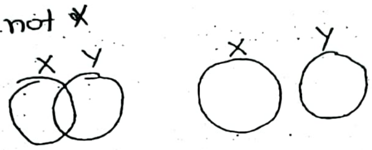
[Invalid]

7. Some X is not Y

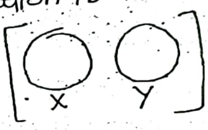


$\exists x \neq y$
$\sim(\forall x = y)$

8. Some Y is not X



Negation - Some X is Y its negation is No X is Y.
$\exists_x(x=y) \Rightarrow \forall_x(x \neq y)$



Ques - P_1: If it rains, the match will not be played
P_2: the match was played.
Which of the following is valid Inference.
$[p \Rightarrow q \wedge \sim q \Rightarrow \sim p]$
(a) it rains
✓(b) it didnot rain
(c) it either rain or didnot rain

Ques P_1: If it rains (p) or one of the team (q) didnot turn up, the match will (r) not be played.
P_2: the match was played.

$p \vee q \Rightarrow r$
$\sim r$
$\therefore \sim(p \vee q)$
$= \sim p \wedge \sim q$

(a) it rain
✓(b) it didnot rain
✓(c) it didnot rain & no team turn up
✓(d) no team turnup.

$\therefore \sim p$
$\sim q$
$\sim p \wedge \sim q$
all are true

Rules of Inference [All these rules are tautology]:

1. Simplification:

$$\frac{p \wedge q}{\therefore p}$$

Ex- if A relation is reflexive then it is accepted
~~it is accepted~~
it is reflexive
∴ it is accepted or symmetric

$$\frac{\begin{array}{l} p \Rightarrow q \\ p \end{array}}{q \Rightarrow [q \vee r]}$$
↳ True

2. Addition:

$$\frac{p}{\therefore p \vee q}$$

3. Conjunction:

$$\frac{\begin{array}{l} p \\ q \end{array}}{\therefore (p \wedge q)}$$

4. Modus Ponens:

$$\frac{\begin{array}{l} p \Rightarrow q \\ p \end{array}}{q}$$

// [Rule of Detachment]

5. Modus Tollens:

$$\frac{\begin{array}{l} p \Rightarrow q \\ \sim q \end{array}}{\therefore \sim p}$$

// [Rule of Contrapositive]
// proof by Contradiction

6. Hypothetical Syllogism:

$$\frac{\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \end{array}}{p \Rightarrow r}$$

// Transitive Rule of Implication

7. Disjunctive Syllogism

$$\frac{\begin{array}{l} (p \vee q) \\ \sim p \end{array}}{q} \qquad \frac{\begin{array}{l} p \vee q \\ \sim q \end{array}}{p}$$

// Proof By Elimination

Ex x>0 or x=0 or x<0
x ≯ 0
x ≮ 0
∴ x=0

8. Constructive Dillema:

$$\frac{\begin{array}{l} p \Rightarrow q \\ r \Rightarrow s \\ p \vee r \end{array}}{\therefore (q \vee s)}$$

9. Destructive Dillema:

$$\frac{\begin{array}{l} p \Rightarrow q \\ r \Rightarrow s \\ \sim q \vee \sim s \end{array}}{\therefore (\sim p \vee \sim r)}$$

PREDICATE LOGIC

Rules of Inference- (for predicate Logic)

1. **Universal Generalization (UG)**: to Prove $\forall x\ P(x)$

$$\left[(P(x_1) \land P(x_2) \ldots \land P(x_i) \ldots) \Rightarrow \forall x\ P(x) \right]$$

Ex- Domain: Jungle animal (LION, Tiger, Cheetah)

p: Lion is wild.
q: Tiger is wild.
r: cheetah is wild.

$\Rightarrow \forall x\ P(x)$: x is wild
$[x \in \text{Domain}]$

2. **Universal specification**-

$\forall x\ P(x)$ is true $\longrightarrow P(x)$

Ex Domain S: {LION, Tiger, Cheetah)

Argument: All Animals are wild $\Rightarrow$ LION is wild
(specification)

Addition: Addition of two integer is integer. $\Rightarrow$ $2+3=5$
5 is integer
(specification)

3. **Existential Quantifier** $\rightarrow$

$P(x_1) \longrightarrow \exists x\ P(x)$

$P(x_1)$: Lion is wild.

$\therefore$ We can say : $\boxed{\text{Some Animal are wild}}$

4. **Existential Specification**-

$\exists x\ P(x) \longrightarrow P(x_1) \lor P(x_2) \lor P(x_3) \ldots$

$\exists x\ P(x)$: Some animals are wild

$\therefore$ We can say - Lion is wild
or
Tiger is wild
or
cheetah is wild
} Specification.

Predicate - It is a propositional function

$P(x): x+2=4$

$\boxed{\begin{array}{l}\forall x\, P(x) \to \text{proposition} \\ \exists x\, P(x) \to \text{proposition}\end{array}}$

Here $\boxed{\begin{array}{l}\forall x\, P(x) - \text{False} \\ \exists x\, P(x) - \text{True}\end{array}}$

Ques - Domain: Set of Real No

1) $P(x): x+0.4=2$

$\therefore \boxed{\exists x\, P(x) \text{ is true}}$

2) $P(x): \sqrt{x}$ is Real

$\boxed{\begin{array}{l}\forall x\, P(x) \text{ is false} \\ \exists x\, P(x) \text{ is true}\end{array}}$

Ques; Domain: set of (+ive integer

$P(x): \sqrt{x}$ is real

$\forall x\, P(x)$ is true $\quad \exists x\, P(x)$ is true

$P(x): \sqrt{x}$ is real $\quad$ on Domain $= Z^+$ $\quad$ on Domain $= R$

$\neg P(x): \sqrt{x}$ is not real

$\forall x\, P(x) \to$ true

$\exists x\, P(x) \to$ true

$\forall x \neg P(x) \to$ false

$\exists x \neg P(x) \to$ false

$\boxed{\begin{array}{l}\forall x\, P(x) \to \text{False} \\ \exists x\, P(x) \to \text{true}\end{array}}$

On two variable -

Domain : Z

$P(x,y) = x+y=10 \quad y = 10-x$

1. $\forall x \forall y\; P(x,y)$ (False)
2. $\exists x \exists y\; P(x,y)$ (True)
3. $\forall x \exists y\; P(x,y)$ (True) $\quad$ 5) $\exists y \forall x\; P(x,y)$ (False)
4. $\exists x \forall y\; P(x,y)$ (False) $\quad$ 6) $\forall y \exists x\; P(x,y)$ (True)

$\exists y \forall x \to \forall x \exists y$

$\exists y$ ↓ constt ↓ for to be true it should be ind. of x

$\forall x$ ↓ Variable

$\forall x$ ↓ variable

$\exists y$ ↓ Variable ↓ can be dependent on x.

Note - if $\forall x \forall y$ is true

$\left.\begin{array}{l}\text{all } \exists x \exists y \\ \exists x \forall y \\ \forall y \exists x \\ \forall x \exists y \\ \exists y \forall x\end{array}\right\}$ are are true

$\left\{\begin{array}{l}\exists y \forall x \equiv \exists x \forall y \\ \forall x \exists y \equiv \forall y \exists x\end{array}\right\}$

↳ for symmetric relation

$\exists \to$ the variable which is with $\exists$ should be independent of other if outside

Note- While checking for the functions, it should check for the dependencies & domain.

$$\boxed{\forall x \forall y \rightarrow \exists y \forall x \rightarrow \forall x \exists y \rightarrow \exists x \exists y}$$

Ques- Domain: Natural No
$P(x,y): x+y=10$

1) $\forall x \forall y$ False
2) $\exists x \exists y$ True
3) $\exists x \forall y \equiv \exists y \forall x$ False
4) $\forall y \exists x \equiv \forall x \exists y$ False (due to domain)

Ques- for Assymetric f^n
Domain: integer $P(x,y): x+y^2=10$

$x = 10 - y^2$
$y = \sqrt{10-x}$

1) $\forall x \forall y$ → False
2) $\exists x \exists y$ → True
3) $\exists x \forall y$ → False (x should be independent of y)
4) $\exists y \forall x$ → False (domain as well as not independent of x)
5) $\forall x \exists y$ → False (as y is not in domain)
6) $\forall y \exists x$ → true (in domain)

Ques Domain: Real No $P(x,y): x+y^2=10$

1. $\forall x \forall y$ - False
2. $\exists x \exists y$ - True
3. $\exists x \forall y$ - False
4. $\exists y \forall x$ - False
5. $\forall x \exists y$ - False
6. $\forall y \exists x$ - true

Ques $P(x,y,z) = x+y+z = 10$. Domain: Integers

$z = 10 - x - y$

1. $\forall x \forall y \forall z$ — False
2. $\forall x \forall y \exists z$ — Solⁿ can depend on x & y both & domain (True)
3. $\forall x \exists z \forall y$ → Solⁿ can depend on x but not on y (False)
4. $\exists z \forall x \forall y$ → False (Not independent of x, y) it will be true $[x+y+z = 10+x+y]$
5. $\exists z \exists y \forall x$ → False (not independent of x)
6. $\exists x \exists y \exists z$ → True
7. $\forall x \exists y \exists z$ → True

In case of two $\exists$, take both existential variable together

$\exists z \exists y \forall x$

$z + y = 10 - x$

[Existential Quantifier variable must be independent of data on right of it & can be depends upon data on left of it.
In case of more than one $\exists$, take all together.]

Note: $P(x,y,z)$: $x^3+y^3=z^3$ on domain: Integer

[No integer solⁿ exist for it]

∴ Nothing will be true.

Predicate Properties:

1. if $\forall x P(x)$ is true than $\exists x P(x)$ is also true.

$$\boxed{\forall x P(x) \rightarrow \exists x P(x)}$$

2. If $\exists x \forall y$ is true than $\forall y \exists x$ is true

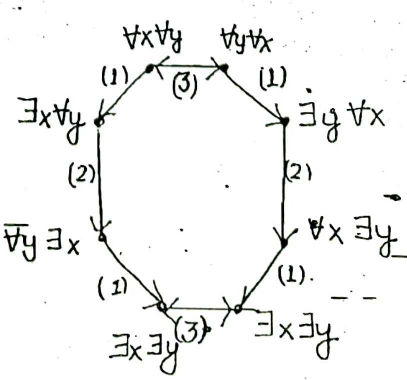
$$\boxed{\exists x \forall y \rightarrow \forall y \exists x}$$ (one way theorem)

$$\boxed{\exists y \forall x \rightarrow \forall x \exists y}$$

3. $\boxed{\forall x \forall y\, P(x,y) \Leftrightarrow \forall y \forall x\, P(x,y)}$

4. $\boxed{\exists x \exists y\, P(x,y) \Leftrightarrow \exists y \exists x\, P(x,y)}$

5.



$\forall x \forall y\, P(x,y) \rightarrow \exists x \exists y\, P(x,y)$ True

$\forall x \forall y\, P(x,y) \rightarrow \forall x \exists y\, P(x,y)$ True

$\forall x \exists y\, P(x,y) \rightarrow \exists x \forall y\, P(x,y)$ False

$\forall x \forall y\, P(x,y) \rightarrow \exists y \exists x\, P(x,y)$ $\boxed{\text{True}}$

$\exists x \forall y\, P(x,y) \rightarrow \exists y \exists x\, P(x,y)$ True

distributive properties

1. $\cancel{\forall x\, (P(x) \wedge Q(x)) \Longleftrightarrow [\forall x\, P(x)] \wedge [\forall x\, Q(x)]}$
2. $\cancel{\exists x\, (P(x) \vee Q(x)) \Longleftrightarrow [\exists x\, P(x)]}$
3. $\cancel{\forall x\, (P(x) \vee (Q(x))) \Longleftrightarrow}$
4. $\cancel{\exists x\, (P(x) \wedge Q(x)) \Longleftrightarrow}$

1. $\forall x\, (P(x) \wedge Q(x)) \Longleftrightarrow [\forall x\, P(x)] \wedge [\forall x\, Q(x)]$
2. $\exists x\, (P(x) \vee Q(x)) \Longleftrightarrow [\exists x\, P(x)] \vee [\exists x\, Q(x)]$
3. $\forall x\, (P(x) \vee Q(x)) \Longleftarrow [\forall x\, (P(x))] \vee [\forall x\, Q(x)]$
4. $\exists x\, (P(x) \wedge Q(x)) \Longrightarrow [\exists x\, P(x)] \wedge [\exists x\, Q(x)]$

To check for the validity

1. $\forall x\, [P(x) \wedge Q(x)]$ Let LHS is true

then $P_1 \wedge Q_1 = T$ $P_1, Q_1 = T$

$\vdots$

$P_n \wedge Q_n = T$ $P_n, Q_n = T$

R.H.S $\Rightarrow$ $\forall x\, P(x) \rightarrow$ True, $\forall x\, Q(x) \rightarrow$ True $\Rightarrow$ True.

- $[\forall x\, P(x) \wedge \forall x\, Q(x)]$ is true

$\therefore$ $P_1 \ldots P_n$ true, $Q_1 \ldots Q_n$ true $\Rightarrow \forall x\, (P(x) \wedge Q(x)] \Rightarrow$ true

Hence it is valid both sides.

2. $\exists x\, (P(x) \vee Q(x))$ Let LHS is true

Let $P_1 = T \ldots P_2 \ldots P_n = F$

$Q_1 \ldots Q_n = F$

R.H.S : $\exists x\, P(x) = T$, $\exists x\, Q(x) = F$ $\Rightarrow$ True

$\therefore$ Hence Valid

• $\exists x\, P(x) \vee \exists x\, Q(x)$ is true than atleast one of $P_1 \ldots P_n$ & $Q_1 \ldots Q_n$ will be true then $\exists x (P(x) \vee Q(x))$ will be true.

Hence, it is valid for both sides.

3.) • Let LHS $[\forall x (P(x)) \vee \forall x (Q(x))]$ is true Let $P_1 \ldots P_n = \text{False}$

$Q_1 \ldots Q_n = \text{True}$

$\therefore \forall x (P(x) \vee Q(x))$ is true.

Hence it is valid

• L.H.S $\forall x (P(x) \vee Q(x))$ is true

Let $P_1 = T$ $P_2 \ldots P_n = \text{False}$

$Q_1 = F \ldots Q_n = F$

$\therefore \forall x\, P(x) \vee \forall x\, Q(x) \Rightarrow \text{False}$

[Hence it is not valid]

4.) • $\exists x (P(x) \wedge Q(x))$ is true

ie $P_1 = T \Rightarrow Q_1 \cdots Q_n = F$ $\Rightarrow$ it is true

$Q_1 = T$ $P_1 \ldots P_n = F$

$\therefore$ $\exists x\, P(x) \Rightarrow$ true, $\exists x\, Q(x) \Rightarrow$ true $\Rightarrow$ true Hence it is valid.

• $[\exists x (P(x))] \wedge [\exists x\, Q(x)]$ is true

$\exists x\, P(x)$ is true Let $P_1 = T$, $P_2 \ldots P_n = F$

$\exists x\, Q(x)$ is true $Q_1 = F$ $Q_2 = T$, $Q_3 \ldots Q_n = F$

$\therefore \exists x (P(x) \wedge Q(x)) \Rightarrow \text{False}$

Hence it is invalid.

<u>Note</u>- 1. $\forall x (P(x) \Rightarrow Q(x)) \Rightarrow [(\forall x\, P(x)) \Rightarrow (\forall x\, Q(x))]$

2. $\exists x (P(x) \Rightarrow Q(x)) \Leftarrow [(\exists x\, P(x)) \Rightarrow (\exists x\, Q(x))]$

1. LH.S : $\forall x\, P(x) \Rightarrow Q(x)$ is true $P_1 \Rightarrow Q_1$

$P_2 \Rightarrow Q_2$

$\vdots$

$P_n \Rightarrow Q_n$

R.H.S : $[\forall x (P(x) \Rightarrow Q(x)) \wedge \forall x\, P(x)] \Rightarrow [\forall x\, Q(x)]$

Hence it is trivially true

[valid]

<u>L.H.S</u>: $[\forall x\, P(x) \Rightarrow \forall x\, Q(x)]$ is true

$[P_1 \ldots P_n \Rightarrow Q_1 \ldots Q_n]$

<u>R.H.S</u> $P_1 \Rightarrow Q_1 \rightarrow$ False as $[P_1 \ldots P_n \Rightarrow Q_1]$

$\therefore$ [Hence it will be invalid]

2. $[\exists x(P(x) \Rightarrow Q(x))] \Rightarrow [\exists x(P(x)) \not\Rightarrow \exists x\, Q(x)]$

L.H.S Let LHS be true

$\therefore \cancel{P_1 = 3 \Rightarrow Q_3}$ $P_3 \Rightarrow Q_3$ $P_3 = T$
$Q_3 = T$

RHS $[\exists x[P(x) \Rightarrow Q(x)] \Rightarrow \exists x\, P(x)]$

$\downarrow P_5 = T$ Since $P_3 \neq P_5$ than Q_x will not exist

Hence it is invalid.

3. $[\exists x\, P(x) \Rightarrow \exists x\, Q(x)] \Rightarrow \exists x(P(x) \Rightarrow Q(x))$

Let LHS be true

$P_5 \Rightarrow Q_6$ $P_5 = T$
$Q_6 = T$

RHS $\Rightarrow$ $P_5 \Rightarrow Q_5 \rightarrow$ May or may not be true

$P_6 \Rightarrow Q_6 \rightarrow$ true

[Hence it is valid]

~~Dat~~ 15-Sept-2017

Properties

1. $\forall x\,(P(x) \wedge Q) \longleftrightarrow [\forall x\, P(x)] \wedge Q$
2. $\exists x\,(P(x) \wedge Q) \longleftrightarrow \exists x\, P(x) \wedge Q$
3. $\forall x(P(x) \vee Q) \longleftrightarrow \forall x\, P(x) \vee Q$
4. $\exists x\,(P(x) \vee Q) \longleftrightarrow \exists x\, P(x) \vee Q$

- 1. $\forall x\,(P(x) \wedge Q) \longrightarrow [\forall x\, P(x)] \wedge Q$

L.H.S $P_1 \wedge Q = T$ $\Rightarrow P_1 \ldots P_n = T$
$P_n \wedge Q = T$ $Q = T$

$\therefore$ RHS is also true.

R.H.S $\forall x\, P_1 \ldots P_n = $ True
$Q \rightarrow$ true

$\therefore$ Hence valid

$\forall x\, P(x) \wedge Q \rightarrow \forall x\,(P(x) \wedge Q)$

$P_1 \ldots P_n = T \wedge Q = T \rightarrow \forall x(P(x) \wedge Q) \Rightarrow$ true

Valid

2. $\forall x(P(x) \vee \varphi) \longleftrightarrow [\forall x P(x)] \vee \varphi$

L.H.S: Let LHS be true

$\varphi \Rightarrow F \quad P_1 \ldots P_n = T \quad \Rightarrow \forall x P(x) = T$

OR

$\varphi = T \quad P_1 \ldots P_{n-1} = T \quad P_n = F \Rightarrow \forall x P(x) = F$

but $\forall x(P(x) \vee \varphi) = T$

— proved

3. $\exists x (P(x) \wedge \varphi) \longleftrightarrow \exists x P(x) \wedge \varphi$

L.H.S → Let L.H.S be true

$\therefore \varphi = T, P_1 = T \quad P_2 \ldots P_n = F$

$\Downarrow$

$\varphi = T \wedge \exists x P(x) = T \quad \Rightarrow$ true

Proved.

4. $\exists x(P(x) \vee \varphi) \longleftrightarrow \exists x P(x) \vee \varphi$

Let L.H.S be true $\quad \cancel{P(x)}\ P_1 = T, P_2 \ldots P_n = F, \varphi \Rightarrow F$

$\exists x P(x) \vee \varphi = T \vee F = T$

Let LHS be true $\quad P_1 \ldots P_{n-1} = F \quad \varphi = F \quad P_n = T$

$\therefore$ H prove $\exists x(P(x) \vee \varphi) = T$

Hence it is valid.

<u>Ques</u>- Check for the validity

$\forall x(P(x) \vee \varphi) \longleftrightarrow \exists x(P(x)) \vee \varphi$

L.H.S: Let LHS be true

$P_1 = T, \varphi = F, P_2 \ldots P_n = F \rightarrow \exists x(P(x)) \vee \varphi = T$ Hence valid

$\boxed{\forall x(P(x) \vee \varphi) \longrightarrow \exists x P(x) \vee \varphi}$ <u>valid</u>

Let LHS is $\exists x P(x) \vee \varphi = T$

$P_1 = T \quad \varphi = F \rightarrow \quad P_1 \vee \varphi \Rightarrow T$

$P_2 \ldots P_n = F \quad P_2 \vee \varphi \Rightarrow F$ Hence it is invalid

$\boxed{\exists x(P(x)) \vee \varphi \longrightarrow \forall x(P(x) \vee \varphi) \rightarrow \text{invalid}}$

<u>Note</u>: 1. $\forall x(P(x) \longrightarrow \varphi) \rightarrow \boxed{\forall x P(x) \longrightarrow \varphi}$

L.H.S: $P_1 \rightarrow \varphi = T$

$P_2 \rightarrow \varphi = T$

$\vdots$

$P_n \rightarrow \varphi = T$.

R.H.S $P_1 \wedge P_2 \wedge \ldots P_n \rightarrow \varphi$

→ <u>valid</u>

<u>Ques</u> $\quad \forall x(P(x) \rightarrow \varphi] \longrightarrow [\exists x P(x) \longrightarrow \varphi]$ → valid

$\forall x(P(x) \rightarrow Q(x)) \longrightarrow [\exists x P(x)] \rightarrow [\forall x(Q(x))]$

invalid

$\exists x (P(x) \rightarrow Q) \longrightarrow [\exists x\, P(x)] \rightarrow [Q]$

L.H.S $P_1 \rightarrow Q = T$ $[0, F] = T$ R.H.S $P_2 \ldots P_n = T \rightarrow Q \Rightarrow F$

$P_2 \rightarrow Q$, ..., $P_n \rightarrow Q$ $\Rightarrow F$ $[1, 0] = F$ F

Hence invalid

$\exists x (P(x) \rightarrow Q) \longleftarrow \{ [\exists x (P(x))] \rightarrow Q \}$

R.H.S ~~L.H.S~~ [true] $P_1 \rightarrow Q = $ True, $P_2 \ldots P_n$ } False

~~R.H.S~~ L.H.S $P_1 \rightarrow Q$ ↳ true

Hence valid

[0, 0] ✓
[0, 1] ✓
[1, 1] ✓
$P_5 = T$ $Q = T$

Translations - ($\exists, \forall$ has higher precedence than everyone)

By default: Domain is set of all object in universe

for No: Set of Real No

1. There exist ($\exists$) - Some ≡ atleast one
2. for all ($\forall$) - Every ≡ all ≡ plural ≡ any

[Domain: Every Student in the Class]

Every student in the Class has studied calculus.

~~P(x): x is the student in the Class~~

~~Q(x): x has studen~~

C(x): x has studied ~~ele~~ calculus

$[\forall x\, C(x)]$

Every student in the class has studied calculus.

2. Every Student in the class has studied calculus. No Domain is given.

S(x): x is the student of class

C(x): x has studied calculus

$\forall x\, [S(x) \rightarrow C(x)]$

$[\forall x\, (S(x) \wedge C(x)) \Rightarrow$ Every object in the universe is student of the class & has studied calculus.

↳ invalid translation

Note- $\forall$ along with and ($\wedge$) may cause the invalid statement, when domain is not specified.

3. Q: All the cars in expressway

1. Every fast car on the expressway is dangerous.

$\forall x\, F(x) \rightarrow D(x)$

F(x): x is fast

D(x): x is dangerous.

2. Every car on the express way is fast and dangerous.

$\forall x (F(x) \land D(x))$

3. Some student in the class has student calculus.

1) No Domain

$\exists x [S(x) \land C(x)]$

S(x): x is the student.
C(x): x has studied calculus.

2. Domain: Every Student in the Class.

$\exists x \, C(x)$

3. $\exists x (S(x) \rightarrow C(x))$ - Some object in universe if student of class, than studied calculus.
It will cause problem when No student in the class exists than also it is true. [Implication never occurs with $\exists$]

- which is $\equiv$ who is $\equiv$ adjective $\equiv$ (and)

There is somebody who is tall.

$\exists x (S(x) \land T(x))$

S(x) → x is somebody.
T(x) → x is tall.

- H(x): x is humming bird
  S(x): x is small
  $\exists x(H(x) \land S(x))$ ⇒ Humming bird is small
  there exist humming bird which is small.

- Domain: All Student of Class
  Every Tall Boy of class plays Basketball.
  $\forall x (T(x) \land B(x) \longrightarrow BB(x))$

  T(x): x is tall
  B(x): x is boy
  BB(x): x plays basketball

- Domain: Jungle
  $\exists x(H(x) \lor S(x))$: Some object exist in jungle which are either humming bird or small.

No and Not -

1. No hardworking ~~poor~~ person is poor. $P(x) \to x$ is poor.
$H(x) \to x$ is hardworking
$R(x) \to x$ is rich.

$\forall x\ H(x) \Rightarrow R(x)$

$\forall x\ [H(x) \Rightarrow \sim P(x)]$

2. Not all Hardworking person are ~~poor~~.

$$\begin{bmatrix} \exists x\ [H(x) \wedge \sim P(x)] \\ \exists x\ [H(x) \wedge R(x)] \end{bmatrix}$$

From brute form method,

$\Rightarrow \sim$ (all Hardworking people are poor)

$\Rightarrow \sim (\forall x\ H(x) \Rightarrow P(x))$

$\Rightarrow \exists x [\sim (\sim H(x) \vee P(x)]$

$\Rightarrow \exists x [H(x) \wedge \sim P(x)]$

Note

$$\boxed{\text{No Hardworking person is poor} \equiv \text{Not (a Hardworking person is poor)}}$$

$\sim [\exists x\ H(x) \wedge P(x)]$

$\forall x (\sim H(x) \vee \sim P(x)]$

$$\boxed{\forall x\ (H(x) \Rightarrow \sim P(x))}$$

Type II - 1. Diamonds and Pearls are precious

$D(x) \to x$ is diamond
$P(x) \Rightarrow x$ is pearls
$Pr(x) \Rightarrow x$ is precious

$\forall x\ (D(x) \wedge P(x) \Rightarrow Pr(x))$

[Every object which is both diamond & pearls is precious.]

↳ wrong

any object is either pearls or diamond.

The correct translation is

$$\forall x\ (D(x) \vee P(x) \Longrightarrow Pr(x))$$

Ques- Lions & tiger Attack when hungry or threaten.

(i) $\forall x\,(L(x) \vee T(x)) \Rightarrow [(H(x) \vee Th(x)) \Rightarrow A(x)]$ (now is present in the sentence)

(ii) $\forall x\,(L(x) \wedge T(x)) \Rightarrow [(H(x) \vee Th(x)) \Rightarrow A(x)]$

[every object which is Lion as well as tiger]

(iii) $\forall x[(L(x) \vee T(x)) \Rightarrow (H(x) \vee Th(x))] \Rightarrow A(x)$

[If Every Lion & tiger is threaten & hungry & than attack]

(iv) $\forall x[(L(x) \vee T(x)) \wedge (H(x) \vee Th(x))] \Rightarrow A(x)$

$p \to (q \to r) \equiv (p \wedge q) \to r$

(v) $\forall x[(H(x) \vee Th(x)) \to (L(x) \vee T(x) \to A(x))]$

left bracket is taken if sentence is contain 'if' in it

$$\left[\begin{array}{lcl} p \to (q \to r) & \equiv & q \to (p \to r) \\ p' + q' + r & \equiv & q' + p' + r \end{array}\right]$$

Ques- All purple mushroom are poisnous.

1. $\forall x\,[P(x) \wedge M(x) \to Po(x)]$
2. $\forall x\,[P(x) \to (M(x) \to Po(x))]$
3. $\forall x\,[M(x) \to [P(x) \to Po(x)]]$

(All are correct)

4. $\forall x\,[(P(x) \to M(x)) \to Po(x)] \to$ incorrect

[Every object which is purple is a mushroom & then poisnous.]

Ques- • Student & parents are anxious about result of xam.

$\forall x\,(S(x) \vee P(x) \Rightarrow Ax(x))$

• Young Student are anxious about result of exam.

$\forall x\,(S(x) \wedge Y(x) \Rightarrow Ax(x))$

Ques- Domain: All the students of college.

$F(x,y)$: x and y are friends [Symmetric f^n]

1. $\forall x \forall y\, F(x,y)$: ~~Every~~ All Students of college is friendly with every other student
2. $\forall x \exists y\, F(x,y) \equiv \forall y \exists x\, F(x,y)$: Every Student in college is friendly with someone.
3. $\exists y \forall x\, F(x,y) \equiv \exists x \forall y\, F(x,y)$: Some students are friendly with every student of college.

4. $\exists x \exists y\ F(x, y)$: There is atleast a pair of. Singer who are joinup.

ues No one can fool everyone all the time.

Domain: x is a person.

$F(x, y)$: x can fool y all the time.

~~$\forall x \forall y$~~ No one can fool everybody all the time

Not a

$\neg(\exists x \forall y\ F(x,y))$

$\equiv \forall x \exists y \neg F(x,y)$

↳[x can not fool atleast one all the time]

**

ues Some boys are taller than all the girls of class

$$\exists x\,(C(x) \wedge B(x)) \wedge [\forall y\,(G(y) \wedge C(y)) \rightarrow T(x,y)]$$

Ques- $F(x,y)$: x has visited website y. Domain x: All student

y: All website

1) $\forall x \forall y$; Every student has visited ~~some~~ every website

$\forall y \forall x$: Every Student has visited by atleast one student.

2. $\forall x \exists y$: Every website has been visited by atleast one student

3. $\forall y \exists x$: Every website has been visited by atleast one student

4. $\exists x \forall y$: Atleast one student who has visited all the website

5. $\exists y \forall x$: There are some websites which are visited by every student.

6. $\exists x \exists y$: Some student has visited some website y

7. $\exists y \exists x$: Some website has been visited by some student.

Domain: Student

1.ues $F(x,y)$: x shows friend to y.

$\forall y \forall x\ F(x,y)$: Everystudent receives friendship from everyone.

1. $\forall x \forall y\ F(x,y)$: Every student shows friendship to everyone

$\forall x \exists y\ F(x,y)$: Every Student of college shows friends to atleast one student

$\exists y \forall x\ F(x,y)$: There is a student who receives frndship from all.

$\exists x \forall y.\ F(x,y)$: There is a student shows frndship to all.

$\forall y \exists x\ F(x,y)$: Every student in college receives frndship from atleast one student.

$\exists x \exists y\ F(x,y)$: Some student shows frnd to someone.

$\exists y \exists x\ F(x,y)$: Some student receives frndship from someone.

- Some painters has painted Every drawing

$P(x,y)$: x has painted y — $x \Rightarrow$ all painter, $y \Rightarrow$ all drawing

$\exists x \forall y P(x,y)$

Some drawing has been painted by every painter.

$\exists y \forall x P(x,y)$

When No domain is given.

1. Some painter has painted all drawing

$$\left[\begin{array}{l} P(x): \text{x is a painter.} \\ D(y): \text{y is drawing.} \\ Pa(x,y): \text{x painted y} \end{array}\right]$$

$\cancel{\exists x \forall y (P(x) \land D(y) \land Pa(x,y))}$

$\exists x (P(x) \land \forall y (D(y) \rightarrow Pa(x,y)))$

$\equiv \exists x \forall y (P(x) \land (D(y) \rightarrow Pa(x,y))$

** $\cancel{\exists x}$ $\forall y (D(y) \rightarrow \exists x (P(x) \land Pa(x,y)))$

False

$\boxed{\text{for every painting there exist one painter.}}$

$\cancel{P \cdot x}$
$P(d' + pa)$
$pd' + p\,pa$
$d'(P \land pa)$
$\cancel{d'P +}$ $d' + p\,pa$

2. Some boys in the class is taller than every girl of class.

$B(x)$: Boy in Class

$G(x)$: Girl in Class

$T(x,y)$: x tall than y

$$\exists x [B(x) \land \forall y (G(y) \rightarrow T(x,y))]$$

$$\equiv \exists x [B(x) \land \forall y (\sim T(x,y) \rightarrow \sim G(y))]$$

$$\not\equiv \exists x \forall y [B(x) \land G(y) \rightarrow T(x,y)]$$

↳ wrong when No boys in the class

When half domain is given.

- Domain $x \Rightarrow$ Boys in the class

Some boys in class is taller than all girl in class

$$\exists x \forall y (G(y) \rightarrow T(x,y))$$

- Domain $x \Rightarrow$ Girls in the class

$$\exists x (B(x) \land \forall y\, T(x,y))$$

$\forall x [x \neq 0 \rightarrow \exists y (xy = 1)]$

For every $x \neq 0$ there exist a y such that $[y = 1/x]$

$\therefore$ Every non-zero real no has a multiplicative inverse.

> ~~x≠1 or~~ $(x \neq 1) \wedge [x = yz \Rightarrow y = 1 \text{ or } y = x]$

x is a prime No (No divisible by 1 or itself)

> ~~∀y[C(y) V ∃x~~

$\forall x [C(x) \vee \exists y (C(y) \wedge F(x, y))]$

C(X): x has computer

F(X): x &y are frnds

Domain: All student in college

[Every student in college has computer or for every student there exist a frnd who has computer]

> $\forall x \forall y \forall z \; (x * y) * z = x * (y * z)$ -

[* is associative]

$\forall x \exists x' \quad x + x' = 1$
$x \cdot x' = 0$

[Every Elmt in the set has atleast one complement.]

If [Complement + Distributive $\Rightarrow$ Unique complement]

$\exists_0 \forall x \; (x + 0 = x)$ = x has one identity element 0.

xactly 1: There is exactly one apple on the table

Domain $\rightarrow$ all object on table.

A(X) $\rightarrow$ x is an Apple.

~~∃x[A(x) ∧ ...]~~

$\exists x [A(x) \wedge \forall y (y \neq x \rightarrow \neg A(y))]$

OR

$\exists x [A(x) \wedge \forall y (A(y) \rightarrow y = x)]$

OR

$[\exists! x \, A(x)]$ OR $[\underbrace{\exists x A(x)}_{\text{atleast 1}} \wedge \underbrace{\forall x \forall y (A(x) \wedge A(y) \Rightarrow x = y)}_{\text{atmost 1}}]$

Atleast one - There is atleast one apple on table

$\exists x\, A(x)$

Atmost one - $\forall y \forall x (A(x) \wedge A(y) \Rightarrow x=y)$

0 apple → $A(x) \wedge A(y) \rightarrow$ false ⇒ [then true]

Hence, zero apples are allowed.

~~Exactly~~ Atmost one - (zero apple) $\vee$ (one apple)

$$\left[[\forall x \sim A(x)] \vee \exists x(A(x) \wedge \forall y (A(y) \rightarrow x=y)) \right]$$

Atleast two - $\exists x(A(x) \wedge \exists y (A(y) \wedge x \neq y))$

$\equiv \exists x \exists y (A(x) \wedge A(y) \wedge x \neq y)$

no need to add $x=y$ as z takes care for it

Exactly two - $\left[(\exists x (A(x) \wedge \exists y (A(y) \wedge x \neq y)) \wedge \forall z (A(z) \Rightarrow z=x \text{ or } z=y) \right]$

$\left[\exists x \exists y (A(x) \wedge A(y) \wedge x \neq y) \wedge \forall z (z \neq x \wedge z \neq y \Rightarrow \sim A(z)) \right]$

Atmost two - $\forall x \forall y \forall z (A(x) \wedge A(y) \wedge A(z) \Rightarrow z=x \text{ or } z=y \text{ or } x=y)$

Exactly two - $\exists x \exists y [x \neq y \wedge \forall z (A(z) \Leftrightarrow z=x \text{ or } z=y)]$

[atleast two objects]

[If Apple(z) → then either equal to x or y. to whichever it became equal it will become apple]

1. Select
2. Arrange : Select

COMBINATORICS

Permutations & Combination.
Summation — Properties of Binomial f^n / Generating f^n
Solution of Reccurence Relation.

Permutation & Combinations:

Ways to select	No Repitition	Unlimited Repitition	Limited Repitition
PerMutation (Arrange, ordered Sequence, words Pwd, numbers Arrange in team A&B)	^nP_r	n^r	$\frac{n!}{n_1!n_2!n_3!\ldots n_r!}$
COMBINATION (Selection, no order, collection Set, subsets, set committee, inter view selection, team selection)	^nC_r	$^{n+r-1}C_r$	[Generating function]

n - No of object
$r \Rightarrow$ object get selected

PerMutation - Constraints -

(i) start with end with
(ii) not ending with
(iii) atleast one Men selected in team A
(iv) together
(v) not together
(vi) atleast 2 are together
(vi) No two object of certain type are not together
(vii) Alternate way of arranging
(viii) circular

- Straight Line, [n people can be permutated in n! way]
- circular, [way to permute n people = $(n-1)!$] [due to circle symmetry]
- n Necklace, [ways $= \frac{(n-1)!}{2}$] [due to symmetry of beads]

No Repitition

1. Ways to put 10 people on 4 chairs?

$^{10}P_4 \Rightarrow {}^{10}C_4 \times 4!$

[Select 4 people] [arrange on 4 chair]

10 9 8 7

2. Ways to put 4 people on 10 chairs? $^{10}P_4$ [by default object are considered as distinct]

Note- $[{}^{n}P_r \quad n<r = 0]$

10 9 8 7

3. No of 5 letters pwd. from English Alphabet when repitition is not allowed?

$^{26}P_5$

26 25 24 23 22

(b) when repitition is allowed?

$\Rightarrow (26)^5$

(c) No repitition but starts with a & end with z?

$[23 \times 22 \times 24]$

(d) contains atleast one a?

$5 * (26)^4$

(e) 26 letters pwd with distinct element = 26!

4. 26 letter pwd from {a, b, c, d, e} $\Rightarrow (5)^{26}$

(f) not starting with a & no repitition?

$\Rightarrow (25)^2 * 24 * 23 * 22$

25 25 24 23 22

Limited Repitition -

How many possible arrangement for MISSISSIPPI?

$$\frac{11!}{4!\,4!\,2!\,1!}$$

M → 1
I - 4
S = 4
P - 2

all S are together? $\frac{8!}{4!\,2!}$

not start with M? total - start with M

$$= \left[\frac{11!}{4!\,4!\,2!} - \frac{10!}{4!\,4!\,2!}\right]$$

Ways to arrange 5 physics book 5 English book & 4 Math books on a shelf ?

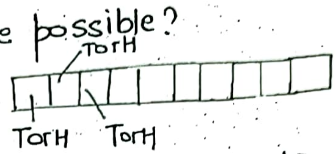
No of book = 14 $\Rightarrow \underline{14!}$ [books by default are not identical]

If books were identical for category $\frac{14!}{5!\,5!\,4!}$

5 ~~ways~~ boys & 6 girls ways to arrange? $11!$

10 coins are tossed, No of outcome possible?

$= (2)^{10}$



10 dices are thrown, No of outcome possible? $(6)^{10}$

10 games are played b/w two team. Result are win, Loose, Tie !

No of outcome possible $= 3^{10}$

for question paper with 60 question & 4 option cand- can either fill it or it

No of answer sheet possible $= 5^{60}$

No of answer key possible $= 4^{60}$

No of correct answer key possible $= 1$.

Jumbles possible for EQUATION ? $8!$

- Vowels should be together? $4! * 5!$

5 letter pwd possible starting with A and M

$(26)^3$

- starting with A or ending with M

$= \begin{bmatrix} \text{Starting with} \\ A \end{bmatrix} + \begin{bmatrix} \text{ending with} \\ m \end{bmatrix} - \begin{bmatrix} \text{starting with } A \\ \& \\ \text{ending with } M \end{bmatrix}$

$= (26)^4 + (26)^4 - (26)^3$

$= 51 * (26)^3$

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \oplus B) = n(A) + n(B) - 2n(A \cap B) \equiv n(A \cup B) - n(A \cap B)$
- $n(A-B) = n(A) - n(A \cap B)$
- $n(B-A) = n(B) - n(A \cap B)$
- $n(A^c \cap B^c) = n(U) - n(A \cup B)$
- $n(A^c \cup B^c) = n(U) - n(A \cap B)$
- $n(A^c) = n(U) - n(A)$

Ques- How many 8 bit string (binary) which starts with 00 or end with 111

A: Start with 00

B: End with 111

$$\boxed{n(A \cup B) = 2^6 + 2^5 - 2^3 = 2^3(8+4-1) = 11 \times 2^3}$$

- Starts with 00 or end with 111 but not both.

$n(A \oplus B) = 2^6 + 2^5 - 2^4 = 2^4(4+2-1) = 5 \times 2^4$

- Start with 00 but not end with 111.

$n(A \oplus B) = 2^6 - 2^3 = 7 \times 2^3$

- neither start with 00 & nor end with 111.

$n(A^c \cap B^c) = 2^8 - 11 \times 2^3 = 21 \times 2^3$

Ques- No possible which donot contain 7 when selected from (100 - 999)?

1 to 9 [not 7]	0 to 9 [not 7]	0 to 9 [not 7]

$\Rightarrow 8 \times 9 \times 9$

$\therefore$ possible No $= 81 \times 8 = 648$

$\therefore$ probablity $= \frac{648}{900} = 0.72$

16-Sept-2017

1. How many per. of word 'equation' contains all vowels together?

(AEIOU) T N Q)

$\therefore$ total object $= 4$

$\therefore$ $\boxed{\text{No of ways} = 4! \times 5!}$

- All consonants are together? No of ways $= 6! \times 3!$

- All vowels & all consonant are together? [1V, 1C]

No of ways $= 2! \times 5! \times 3!$

(d) not contains all the vowels together,

$\Rightarrow$ no vowel together $= \neg$(all vowels together)

$= 8! - 4! \times 5!$

(e) atleast two vowel are seperated?

$\equiv$ No vowel are together

(f) atleast two vowel are together?

$= 8! -$ No two vowels are together

≡ permutation of MISSISSIPPI

a) all S together?

M I (SSSS) I I P P I $= \frac{8!}{4!2!}$

• atleast 2 SS are ~~together~~ seperated

$\equiv$ No SS are together

$\equiv \frac{11!}{4!4!2!} - \frac{8!}{4!2!}$

• all M, S, I, P appears together?

$= 4!$

Q- Binary string from 5 0 & 6 1's possible where all zero are together

00000 111111

$\Rightarrow$ 7!

Ques - from the possible jumbles of MISSISSIPI, If one is picked at random what is the probability all S are together?

$$= \frac{\frac{8!}{4!2!}}{\frac{11!}{4!4!2!}}$$

Ques from ~~Eqn~~ ~~No.~~ Word EQUATION,

a) No two consonants are together?

_ E _ U _ I _ A _ O _

$= {}^6P_3 \times 5!$ [in 6 gaps, ways to put 3 consonants]

$\downarrow$

arranging of vowels

(b) No two vowels are together?

$\_ O \_ T \_ N \_$ ⇒ Not possible

≡ 0 ways

Ques 5 Girls, 4 Boys ____

Ways

1) No two Boys sit together? $5! \times {}^6C_4$

2) No two girls ________ ? $4! \times {}^5P_5$

· Arrange the object which is other from on which condn is imposed

Ques m 0's & n 1's such that no two 1's are together?

~~m+1~~ $1 \times {}^{m+1}C_n$

↓ Ways to arrange 0

↳ ${}^{m+1}C_n$

↳ due to repition C is taken (order does not make any difference)

0 _ 0 _ 0 _ 0 . . . 0 _

Ques In MISSISSIPPI, No two S are together?

Arrange other thing than S, ways = $\frac{7!}{4!\,2!}$

Ways to put S = ${}^8C_4$ [for 7 objects, gaps will be 8]

$$\boxed{\text{Ways} = \frac{7!}{4!\,2!} \times {}^8C_4}$$

Ques- Ways to arrange 5B & 6G such that two particular girls want to sit together

5B + 5G = $10! \times 2!$

[two girls] taken together

(b) two particular girls never sit together-

$= 11! - 10!\,2!$

~~Ques~~- Note- $\boxed{\text{1) Alternating Arrangement is possible both object are equal or 1 extra.}}$

Ques- 5B & 5G ways to arrange them in alternating way)

$5! \times 5! \times 2!$

↳ when boy at even place

↳ when boy at odd place

$[2 \times n! \times n!]$

- 5B & 6G for arranging them alternating.

$5! * 6!$

No of Binary string possible with n 0's & n 1's when placed alternating
$= 1 * 1 * 2$ $\quad [{}^nC_n * {}^nC_n * 2]$

- (n+1) 0's & n 1's

0 1 0 1 0 1 0 1 0 $\quad$ 4 1's & 5 0's

$= 1$ way

- Arrangement for circular table?

1) 5 Boys on circular table?

$\frac{5!}{5} = 4!$

2) 6 Boys & 7 girls around circular table?

$12!$

3) 6 Boys & 7 girls around circular table all girls together?

6 Boys + 1 obj = 7 $\Rightarrow \frac{7!}{7} * 7! = 6! * 7!$

4) 6 Boys & 7 girls around circular table such that particular two girls sit together?

6 B + 6 Girls (5 + 2)
single

$= 12 \rightarrow \frac{12! \times 2!}{12} = 11! * 2!$

5) 6B & 7 girls No two ~~boys~~ girls sit together?

Not possible

6. 6B & 7G No two boys sit together

$\frac{7!}{7} * {}^7P_6 = [6! * {}^7P_6]$ $\quad$ [There will not be any diff of even & odd place due to circle symmetry]

7. n boys & n girls alternating?

$[(n-1)! \times n!]$

* How many Necklace possible with 5R, 6G, 3Y circullar beads? [not identical]

total objects = 5+6+3 = 14

$\therefore$ $\boxed{\text{Ways} = 13!/2}$ [due to symmetry & direction]

- 5 idential Red, 6 identical Green, 3 identical Yellow beads?

$$\frac{14!}{5!\,6!\,3!\times 14\times 2} = \frac{13!}{5!\,6!\,3!\times 2}$$

14 → [to make it circular]

2 → [to manage the direction]

- First arrange straight
- than circular
- than direction.

1. How many 4 digit even No are there?

9 10 10 5 ⇒ <u>4500</u>

- 4 digit odd No

9 10 10 5 ⇒ 4500

[1 - 9] (0-9) (0-9) (1,3,5,7,9)

- 4 digit odd no with distinct digit.

8 8 7 5 = 8*8*7*5

(1,3,5,7,9)

- 4 digit ~~odd~~ even No with distinct digit.

9 8 7 (0,2,4,6,8) + 8 8 7 4

$\boxed{\text{No of digits} = 9*8*7*1 + 8*8*7*4}$

$= 504 + 1792$

$= \underline{2296}$

Ques- Ways to make 5 letter pwd with atleast one a?

password = total password - with no a

$= (26)^5 - (25)^5$

- exactly 1 a

$= 5*(25)^4$

• atmost 1a ?

= zero a's + 1 a's

$$\left[\text{Ways} = (25)^5 + 5 \times (25)^4\right]$$

• atleast 2a ?

$$= \underset{\text{all ways}}{(26)^5} - \left[\underset{0\text{ a's}}{(25)^5} + \underset{1\text{ a's}}{5 \times (25)^4} + \underset{2\text{ a's}}{{}^5C_2 \times (25)^3}\right]$$

10 letter words possible with distinct letter containing 4 vowels & 6 consonants

$$ {}^{10}C_4 \times \underset{4V}{{}^5C_4} \times \underset{6\text{consonants}}{{}^{21}C_6} \times 6! \times 4! = 10! \times {}^5C_4 \times {}^{21}C_6$$

Combination

1. Select 5 people from committee of 10 people? $^{10}C_5$

2. Ways to select 3 people from 10 people for three different pa ?

$$^{10}C_3 \times 3! = {}^{10}P_3$$

3. 8 bit binary no with exactly 3 0's?

$\underline{^8C_5}$ ☐☐☐☐☐☐☐☐ ↳ select 5 place to put 1

• atleast 3 0's

$$= \underset{\text{all comb.}}{2^8} - \underbrace{\left[{}^8C_0 + {}^8C_1 + {}^8C_2\right]}_{\text{atmost 2 0's}}$$

• atmost 3 0's

$$= {}^8C_0 + {}^8C_1 + {}^8C_2 + {}^8C_3$$

• atleast 1 0's

$$= 2^8 - 1$$

• probability of containing atmost 2 zero's ?

$$= \frac{{}^8C_0 + {}^8C_1 + {}^8C_2}{2^8}$$

Selection from Multiple group:

1) from 8 B & 5G Select 3 Boys & 1G ?

$^8C_3 * {^5C_1}$

2. from 8B & 5G Select 3 Boys or 1G

$^8C_3 + {^5C_1}$

3.) ways to select from 52 cards, 5 card. exactly 3 is K & 2 are Q.

52
4K | 4Q
5
3K | 2Q

$= {^4C_3} * {^4C_2}$

[HyperGeometric Distribution]

4) Selecting 5 cards that exactly 3 is K & 1 is Queen.

$[{^4C_3} * {^4C_1} * {^{44}C_1}]$

King, Queen, Remaining other card

5. ~~8B, 5G Select 3~~

8G, 5B select 3G & 2B.

- probability exactly 3 G & exactly 2B. $\Rightarrow \left[\frac{^8C_3 * {^5C_2}}{^{13}C_5}\right]$

13
8G | 5B
5
3G | 2B

- choose 5 people with atleast 3 girls.

5
3G | 2B
4G | 1B
5G | 0B

$\Rightarrow {^8C_3} * {^5C_2} + {^8C_4} * {^5C_1} + {^8C_5} * {^5C_0}$

- 8G 1B choose 5 people and atleast 3 girls

9
8G | 1B
5
~~3~~ | 2 → NA
4 | 1
5 | 0

$\Rightarrow [{^8C_4} * {^1C_1} + {^8C_5}]$

- ~~atleast~~ most one Boy & atleast 0 girl

13

8   5   5

~~3   2~~ Not valid

4   1

5   0

$$\left[ways = {}^{8}C_{4} * {}^{5}C_{1} + {}^{8}C_{5} * {}^{5}C_{0}\right]$$

- atleast one Boy & atleast 3 girls

13

8G   5B

5B

3   2

4   1

~~5   0~~

$$\Rightarrow {}^{8}C_{3} * {}^{5}C_{2} + {}^{8}C_{4} * {}^{5}C_{1}$$

- Select 5 people with 3 Girls & 2B that a particular girl is not included in any committee.

$${}^{7}C_{3} * {}^{5}C_{2}$$

- a particular girl in every committee

$${}^{7}C_{2} * {}^{5}C_{2}$$

- a particular girl & a particular boy in every committee

$${}^{7}C_{2} * {}^{4}C_{1}$$

- a particular girl & a particular boy didnot work together?

All committee − they work together

$$= {}^{8}C_{3} * {}^{5}C_{2} - {}^{7}C_{2} * {}^{4}C_{1}$$

$$\left[{}^{7}C_{2} * {}^{4}C_{2} + {}^{7}C_{3} * {}^{4}C_{1}\right]$$

↓

[don't take ${}^{11}C_{3}$]

Ques- Consider the set $\{1, 2, 3 \dots 10\}$ ways to select & arrange 3 No

$${}^{10}P_{3} = ({}^{10}C_{3} * 3!)$$

ways to select & arrange 3 No in ascending order.

$$= [{}^{10}C_{3} * 1] \text{ ways}$$

Combination with Unlimited Repitition:

• In this Situation $[r > n \text{ or } r \leq n]$

ways to select 3 letters $\{a, b, c, d, e, f\}$

- Repitition is also allowed
- $\neq {}^6C_3$
- ∵ we cant to define r, n uniquely.

∴ ways $x_a + x_b + x_c + x_d + x_e + x_f = 3$

[x_a, ⇒ selected a ie no of selected a's]

$$\left[x_1 + x_2 + x_3 \dots x_n = r\right]$$

x_n ↓ No of objects; r ↓ [selected elmt]

∴ Ways $= {}^{6-1+3}C_3 = {}^8C_3$ [when unlimited repitition are allowed]

1) Choclate Problem: Three jars filled with idential Red, Green, Blue choclate

G B R

How many ways to select 10 choclate from these 3 jars

$x_G + x_B + x_R = 10$

• selection of 10 letter from a set $\{a, b, c\}$

[∴ Ways $= {}^{3+10-1}C_{10} \Rightarrow {}^{12}C_{10} = 66$ ways]

• selection of coins (1Rs, 2Rs, 5Rs, 10Rs) to make sum of 100

$x_1 + x_2 + x_3 + x_4 = 100$

$$\left[{}^{103}C_{100}\right]$$

2) Non-Negative Integral Solution: for $[a \geq 0, b \geq 0, c \geq 0]$ → non negative

$a + b + c = 10$

Here $\left[\begin{matrix} n = 3 \\ r = 10 \end{matrix}\right]$

∴ No of $\text{sol}^n = {}^{10+3-1}C_{10} = {}^{12}C_{10}$

3. Identical Ball in the Box problem:

- Distribution problem
- distribute 10 identical balls into 3 boxes

· It is reduced to non-negative integral sol" problem.

$$x_1 + x_2 + x_3 = 10$$

x_1 → no of balls in 1st box; x_2 → balls in 2nd box; x_3 → balls in 3rd box $\Rightarrow \left[{}^{12}C_{10}\right]$

1. Outcome of identical coin tosses & identical dice:

↳ for this HT is same as TH

→ No of Head & No of tails matter only, due to identity

$$X_H + X_T = 10 \Rightarrow {}^{11}C_{10}$$

X_H → No of heads; X_T → No of tails; 10 → No of coins

10 identical dice are tossed-

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10 \Rightarrow {}^{15}C_5$$

x_1 → No of 1's; x_2 → No of 2's; ...; x_6 → No of 6's

; DICE SUM & Digit Sum problem:

3 dice are tossed, ways that total of 3 dice be 12.

$$x_1 + x_2 + x_3 = 12 \qquad \text{Here, } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

x_1 → what come on 1st dice

$$\left.\begin{array}{l} \therefore x_1 \Rightarrow 1 \leq x_1 \leq 6 \\ x_2 \Rightarrow 1 \leq x_2 \leq 6 \\ x_3 \Rightarrow 1 \leq x_3 \leq 6 \end{array}\right\} \left[\text{for a dice}\right]$$

[So variation is needed.]

Digit Sum problem

but if $[x_1 + x_2 + x_3 = 6]$

↳ No upper constraint is needed in it. Since, it can't go more than 6.

To adjust the lower constraint

$$x_1 \geq 1 \quad x_2 \geq 1, \; x_3 \geq 1$$

$$\therefore x_1 - 1 \geq 0 \quad x_2 - 1 \geq 0 \quad x_3 - 1 \geq 0$$

$$\underbrace{x_1 - 1}_{\text{Let } x_1} + \underbrace{x_2 - 1}_{\text{Let } x_2} + \underbrace{x_3 - 1}_{\text{Let } x_3} = 6 - 3$$

$$\left[x_1 + x_2 + x_3 = 3\right]$$

• Upper constraint is satisfied by Generating f^n

Digit Sum problem-

How many 3 digit No whose sum of digit equals to 10

$$x_1+x_2+x_3=10$$

$$1\leq x_1\leq 9 \quad 0\leq x_2,x_3\leq 9$$

- can be done by Generating fn.

3 digit No whose sum of digit equals to 8

$$x_1+x_2+x_3=8 \quad x_1,x_2,x_3\geq 0$$

- No need to satisfy upper constraints

$$1\leq x_1\leq 9 \quad 0\leq x_2,x_3\leq 9$$

$$x_1+x_2+x_3=7$$

$$x_1\geq 0,\ x_2,x_3\geq 0$$

$$\therefore \text{ways} = {}^{7+3-1}C_9 = {}^9C_7$$

Ques- 3 digit string whose sum less than 9

$$x_1+x_2+x_3=9 \quad x_1,x_2,x_3\geq 0$$

$$\boxed{\text{No of string possible} = {}^{9+3-1}C_9 = {}^{11}C_9}$$

~~Variatst~~

Variation-

1. 10 identical balls in 3 boxes- ${}^{10+3-1}C_{10} \Rightarrow {}^{12}C_{10}$

$$x_1+x_2+x_3=10$$

2. 10 identical balls in 3 boxes s.t each box contains atleast 1 ball.

$$x_1,x_2,x_3\geq 1 \quad x_1+x_2+x_3=10$$

$$x_1+x_2+x_3=7 \quad x_1\geq 0,\ x_2\geq 0,\ x_3\geq 0$$

$$\boxed{\text{ways} \Rightarrow {}^9C_7}$$

3) 10 id. ball in 3 boxes s.t 1 box contain atleast 1, 2nd box contain atleast 2 & 3rd box contains atleast 3 ball

$$x_1\geq 1,\ x_2\geq 2,\ x_3\geq 3$$

$$x_1+x_2+x_3=10$$

$$\Downarrow$$

$$x_1+x_2+x_3=4 \quad x_1,x_2,x_3\geq 0$$

$$\text{Ways} = {}^6C_4$$

n id. balls in K box s.t each box contain ≥ 1 ball

$$k_1, k_2, k_3 \ldots \geq 1$$

$$k_1 + k_2 + k_3 + \cdots k_k = n$$

$$\Downarrow$$

$$k_1 + k_2 + k_3 + \cdots + k_k = n - k \qquad k_1, k_2, k_3 \ldots k_k \geq 0$$

$$\therefore \text{Ways} = {}^{n-k+k-1}C_{n-k} = {}^{n-1}C_{n-k} \equiv {}^{n-1}C_{k-1}$$

Note $\boxed{{}^{n}C_{r} = {}^{n}C_{n-r}}$

Ques How many (+)ve integral solⁿ to this equⁿ

$$a+b+c+d = 10 \qquad \underbrace{a \geq 1, b \geq 1, c \geq 1, d \geq 1}_{\text{for (+)ve integral}}$$

$$a+b+c+d = 6$$

$$\text{No. of sol}^n = {}^{9}C_{6}$$

Ques- Ways to sell 10 choclate from 3 types if atleast 1 R & 2 Green must be taken?

$$x_1 + x_2 + x_3 = 10 \qquad x_1 \geq 0, x_2 \geq 1, x_3 \geq 2$$

$$\Rightarrow x_1 + x_2 + x_3 = 7 \qquad x_1, x_2, x_3 \geq 0$$

$$\text{ways} = {}^{9}C_{7}$$

• Outcome for dice two of them atleast 2 & 3 with 6.

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10 \qquad x_2 \geq 2$$

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 5 \qquad x_6 \geq 3$$

$$\left[\text{Ways} = {}^{10}C_{5}\right]$$

• Solⁿ for the eqaⁿ if $x_1 + x_2 + x_3 = 10$ s.t $x_1 \geq -2, x_2 \geq -1, x_3 \geq 1$

$$x_1 + x_2 + x_3 = 10 \qquad x_1 + 2 \geq 0, x_2 + 1 \geq 0, x_3 - 1 \geq 0$$

$$\Downarrow$$

$$x_1 + 2 + x_2 + 1 + x_3 - 1 = 10 + 2 + 1 - 1$$

$$x_1 + x_2 + x_3 = 12 \qquad [x_1, x_2, x_3 \geq 0]$$

$$\left[\text{Ways} = {}^{14}C_{12}\right]$$

3.) $x_1 + x_2 + x_3 = 10$

For 3 jars, you can take atmost 5 choclate from x_1?

Usually upper constraint are satisfied by G.f but we can satisfy it by when constraint is only at one variable by complementary counting.

$x_1 \le 5,\ x_2 \ge 0,\ x_3 \ge 0$

$\Downarrow$ complement from x_1

$x \ge 6,\ x_2 \ge 0,\ x_3 \ge 0$

$\therefore\ x_1 + x_2 + x_3 = 4$

$\Rightarrow$ ways $\Rightarrow$ ${}^{6}C_4$

if Given Jar 1 has only 5 choclate & Jar 2 has only 3 choclate & jar 3 has 5 choclate

$x_1 + x_2 + x_3 = 20$

$x_1 \le 5 \quad x_2 \le 3$

$x_3 \ge 0$

In this, upper bound occurs

$\therefore$ total ways $[x_1 \le 5]$ = All ways − ways $(x_1 \ge 6)$

$= {}^{3-1+10}C_{10} - {}^{6}C_4 = {}^{12}C_{10} - {}^{6}C_4$ Ans

if upper constraint is on two variable-

$x_1 \le 5,\ x_2 \le 3,\ x_3 \ge 0$

$\Downarrow$

complement

$[x_1 \ge 6$ or $x_2 \ge 4]\ x_3 \ge 0$

↳ it is difficult to calculate it

Hence, we will prefer to use generating function.

Special case 1. $x_1 \le 3,\ x_2 \le 20,\ x_3 \le 20$

$x_1 + x_2 + x_3 = 10$

the condition on x_2, x_3 become trivially satisfied. Hence No need to impose them

Variation-

- To put atmost 10 balls in 3 box?

$x_1 + x_2 + x_3 \le 10$

to solve it all cases $[0 \ldots 10]$ all cases will be considered

To do it, add 1 more box in the problem

$x_1 + x_2 + x_3 + x_4 = 10$

when $x_1 + x_2 + x_3 = 0$

make $x_4 = 10$ & so on

$\boxed{\text{ways} = {}^{13}C_{10}}$

ariation- $x_1 + x_2 + x_3 \geq 10$.

[Infinite Sol^n]

ariation- Ways to distribute 10 identical Red ball & 15 identical blue ball & 12 identical Green ball. in 3 boxes.

$$x_1 + x_2 + x_3 = 10R$$
$$= 15B$$
$$= 12G$$

$\therefore$ for Red Ball $= {}^{12}C_{10}$

Green Ball $= {}^{14}C_{12}$

Blue Ball $= {}^{17}C_{15}$

[$\therefore$ total ways $= {}^{12}C_{10} * {}^{14}C_{12} * {}^{17}C_{15}$]

Ques- 2 Girls have picked 10 daffodils, 15 lillies, 12 ~~lillies~~ Sunflower what is the no of ways to distribute them to both?

$$x_1 + x_2 = 10D$$
$$= 15L$$
$$= 12S$$

[Total ways $= {}^{12}C_{10} * {}^{17}C_{15} * {}^{14}C_{12}$]

If daffodills, lillies & sunflower are not unique.

[Total ways $= 2^{12} * 2^{15} * 2^{12}$]

- Distribution Problem
  - Distinct object
    - Distinct group [Partition problem]
    - Identical group [Unordered Partition]
  - Identical Object
    - Distinct group [Unlimited Repitition ${}^{n+r-1}C_r$]

[Combination with unlimited repitition]

Proof- Identical Element to distinct group
$= {}^{n-1+r}C_r$

Suppose r balls are to be placed in n boxes

0 0 0 | 0 0 0 | 0 ... 0 (r balls) ${}^{n-1+r}C_r$

to put them into n boxes $x_1 + x_2 + x_3 \ldots + x_n = r$

Needed to be divided into ~~3~~ $(n-1)$ groups

So, put ~~2~~ $(n-1)$ 1's in these r 0's

∴ Length of string $= r-1+n$

$$\text{ways} = {}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$$

i.e. [r zeros to be placed in a string of length $n+r-1$]

Ques ** 10 books are needed to be arranged on 3 shelf.

$$x_1 + x_2 + x_3 = 10$$

$$\underbrace{{}^{10+3-1}C_{10}}_{\text{way to select place to put 1}} \times \underbrace{10!}_{\text{arrangement of books}}$$

$0_1\ 0_2\ 0_3 \mid 0_4\ 0_5\ 0_6 \mid 0_7\ 0_8\ 0_9\ 0_{10}$

Now 2 1's will be placed. No of objects = 12

$$\frac{12!}{2!} = {}^{n-1+r}P_r$$

(1 will be identical)

Ques- 10 ring way to wear them into 5 finger of left hand =

$R_1 \ldots R_2 \ldots R_3 \ldots R_{10}$

$f_1 \ldots f_4 = 1$ ∴ ${}^{10+5-1}P_{10} = {}^{14}P_{10}$

- for 10 identical rings ${}^{10+5-1}C_{10}$

Distribution Problem

- When objects are distinct-

Distinct → Distinct (Distribute)
- ordered pair/partition

Distinct → Identical (Divide)
- division into groups
- Unordered pair/partition

Distribute 10 toys to 2 child

$(\{t_1, t_2 \ldots t_5\} \{t_6 t_7 \ldots t_{10}\})$

No ordering inside the group

ordering is done outside the group

Divide 10 toys to two groups

$\{\{t_1, t_2 \ldots t_5\} \{t_6 t_7 \ldots t_{10}\}\}$

No ordering inside the group

No ordering outside the group

- ordered partition-

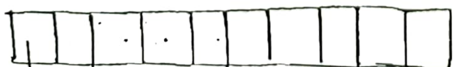
1. Distribution is fully specified,
   Distribution of 10 toys to 3 child each with 3, 3, 4
2. Distribution is fully Unspecified.
   10 toys to 3 child
3. Distribution is partially Specified.
   10 toys to 3 child each get atleast 1.

- Distribute 10 toys to 3 child such that A get 3 toys B get 3 toys C get 4 toys

10 → A: 3, B: 3, C: 4

$$\left[{}^{10}C_3 * {}^{7}C_3 * {}^{4}C_4 = \frac{10!}{3!3!4!}\right]$$

- Distribute 10 toys to 3 child



A or
B or
C or

$$\left[\text{No of Ways} = (3)^{10}\right]$$

Distribution of 5 jobs to 10 people
$=(5)^{10}$

- Distribution is partially Specified:

Ques- Distribute 5 toys that each of 3 child get atleast 1.

	5	
3	1	1
1	3	1
1	1	3
2	2	1
2	1	2
1	2	2

$\frac{5!}{3!1!1!} * 3$

$\frac{5!}{3!} * 3 + \frac{5!}{2!2!} * 3$

Ques- Distribute 20 toys that A get atleast 1 toys

$\therefore$ A get atleast 1 toys = total way − A get 0 toys

$= (3)^{20} - (2)^{20}$

- A get atleast 2 toys

A get atleast two toys = total way − A get 1 toy − A get 0 toys

$= 3^{20} - 2^{20} - {}^{20}C_1 * 2^{19}$

$= \left[3^{20} - 2^{20} - 20 * 2^{19}\right]$

- A get atleast 3 toys

= total way − [A get 0 toy + A get 1 toy + A get 2 toys]

$= 3^{20} - \left[2^{20} + {}^{20}C_1 * 2^{19} + {}^{20}C_2 * 2^{18}\right]$

- A get atmost 2 toys

= A get 0 toy + A get 1 toy + A get 2 toys

$= 2^{20} + {}^{20}C_1 * 2^{19} + {}^{20}C_2 * 2^{18}$

Unordered Partition [fully specified]

4 {a b c d}

$\frac{4!}{2!\,2!} = 6$

ab	cd
ac	bd
ad	bc
bc	ad
bd	ac
cd	ab

$10 \;\; \frac{6}{2} = 3$ ... 5

- Divide 10 people into group of 5 each

$$\frac{10!}{5!\,5!\,2!}$$

[because both group look similar.]

- Divide 10 people into group of 4 & 6. $\frac{10!}{4!\,6!}$

- Divide 10 toys in 2 group of 2 toys each & 2 group of 3 toys.

$$\frac{10!}{2!\,2!\,3!\,3!\,2!\,2!} = \frac{10!}{(2!)^4(3!)^2}$$

- Divide 9 toys in 3 group of 3 toys each

$$= \frac{9!}{(3!)^3\,3!}$$

- Divide 2t objects.

a) 2 group of t object each $= \frac{(2t)!}{(t!)^2\,2!}$

b) t group of 2 object each $= \frac{(2t)!}{(2!)^t\,t!}$

- How many ways to select 22 people out of 40 people for cricket team of 11 each

$$= {}^{40}C_{22} * {}^{22}C_{11} * {}^{11}C_{11} * \frac{1}{2!}$$

$$= {}^{40}C_{22} * \frac{22!}{(11!)^2\,2!}$$

$$= \frac{{}^{40}C_{11} * {}^{29}C_{11}}{2!}$$

40 → 22 → 11, 11

$\frac{40}{22!\,18!} \cdot \frac{22!}{11!\,11!\,2!}$

$= \frac{40}{11!\,\cancel{29!}} \cdot \frac{\cancel{29!}}{11!\,18} \cdot \frac{1}{2!}$

Inclusion Exclusion Principle

- $n(A\cup B) = n(A) + n(B) - n(A\cap B)$
- $n(A\oplus B) = n(A) + n(B) - 2n(A\cap B)$
- $n(A-B) = n(A) - n(A\cap B)$
- $n(B-A) = n(B) - n(A\cap B)$
- $n(A^c\cap B^c) = n(U) - n(A\cup B)$
- $n(A^c\cup B^c) = n(U) - n(A\cap B)$

No b/w 1 to 100 which divisible by 2 or 3.

$n(\text{divisible by 2 or 3}) = n(\text{divisible by 2}) + n(\text{divisible by 3}) - n(\text{divisible by 2 \& 3})$

$= \left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{3}\right\rfloor - \left\lfloor\frac{100}{LCM(2,3)}\right\rfloor$

$= 50 + 33 - 16$

$= 83 - 16 = \underline{\underline{67}}$

$n\left(\text{divisible by 3} \oplus \text{divisible by 2}\right) = 67 - 16 = 51$

$n(\text{divisible by 2 but not 3}) = 50 - 16 = 34$

$n(\text{divisible by 3 but not 2}) = 33 - 16 = 17$

$n(\text{not divisible by 2 \& 3}) = 100 - 67 = 33$

$n(A^c\cup B^c) = 100 - 16 = 84$

No b/w 100 to 1000 both inclusive divisible by 2 or 3:

$n(U) = 901$

- from A to B - includes A & B
- b/w A & B - excludes A & B

$\therefore$ No divisible by 2 or 3 b/w 100 & 1000 = No divisible by 2 or 3 up to (1 to 1000) − No divisible by 2 or 3 (1 - 99)

$= \left\lfloor\frac{1000}{2}\right\rfloor + \left\lfloor\frac{1000}{3}\right\rfloor - \left\lfloor\frac{1000}{6}\right\rfloor - \left\lfloor\frac{99}{3}\right\rfloor - \left\lfloor\frac{99}{2}\right\rfloor + \left\lfloor\frac{99}{6}\right\rfloor$

$= 500 + 333 - 166 - 33 - 49 + 16$

$= 849 - 248 = 601$

For 3 variable -

$$\left[n(A\cup B\cup C) = n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C)\right]$$

$$n(A\oplus B\oplus C) = n(A)+n(B)+n(C)-2n(A\cap B)-2n(B\cap C)-2n(A\cap C)+3n(A\cap B\cap C)$$

→ in 1 - 1000 divisible by 2, 3 or 5 :

$$n[A\cup B\cup C] = \left\lfloor\frac{1000}{2}\right\rfloor + \left\lfloor\frac{1000}{3}\right\rfloor + \left\lfloor\frac{1000}{5}\right\rfloor - \left\lfloor\frac{1000}{6}\right\rfloor - \left\lfloor\frac{1000}{15}\right\rfloor - \left\lfloor\frac{1000}{10}\right\rfloor + \left\lfloor\frac{1000}{30}\right\rfloor$$

$$= 500+333+200-166-66-100+33$$

$$= 1066 - 432$$

$$= \underline{634}$$

$$n(A\oplus B\oplus C) = 500+333+200-332-132-200+99$$

$$= 1066+66 - 664$$

$$= 1132 - 664 = 468$$

$$\left[n(A\cup B - C) = n(A\cup B) - n((A\cup B)\cap C) = n(A\cup B) - n((A\cap C)\cup(B\cap C))\right]$$

Problem of Derangement

No of possible ~~deg~~ Derrangement = D_n.

$n \Rightarrow$ No of element in the Set

if $A = \{1,2,3,4,5\}$

No of ~~Deg~~ Derrangement = D_5

Derrangements - Way to arrange the No. such that no number is present at its position.

$$[3\ 1\ 2\ 5\ 4]$$

~~$n(A\cap B\cap C\cup D\cup E) = D_5$~~ ~~A → A is not at its pos^n~~

~~B → B is not at its pos^n~~

~~C → C is not at its pos^n~~

~~Der~~

Basically $\boxed{D_n = \sum_{r=2}^{n} (-1)^r \frac{n!}{r!}}$

$\therefore D_5 = \sum_{r=2}^{5} (-1)^r \frac{5!}{r!}$

$= \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5!}{5!}$

~~n(AUBUCUDUE)~~

<u>Ques</u>- Consider 5 No boxes & 5 no ball (numbered). What is the probability that Ball i is ~~in~~ not in the ith box for i = 1, 2, 3, 4, 5?

A- D_5 [Problem of Derrangement]

- way such that Ball i is in the ith box for i=1,2,3,4,5?
  $=1$. $\Rightarrow$ probability $= 1/120$
- every ball in the wrong box? D_5 $\Rightarrow$ probability $= 119/120$
- atleast one is in the right box
  $= 5! - D_5$ $\Rightarrow$ probability $= 1\cancel{19}/120$
- atleast one is in the wrong box
  $= n! -$ Every ~~Balls~~ one is in the correct box
  $= 5! - 1$ $\Rightarrow$ probability $= 119/120$

<u>Proof</u>: 5 numbered boxes with 5 numbered ball.

A: ball 1 is in ~~ball~~ box 1

B: ball 2 is in box 2

$\therefore D_5 = n(A^c \cap B^c \cap C^c \cap D^c \cap E^c) = n(U) - n(A \cup B \cup C \cup D \cup E)$

$= n(U) - [n(A) + n(B) + n(C) + n(D) + n(E) - n(A\cap B) - n(A\cap C) - n(B\cap C) - \dots + n(A\cap B\cap C) + \dots - n(A\cap B\cap C\cap D) + n(A\cap B\cap C\cap D\cap E)]$

$\therefore n(A) =$ Ball 1 in box 1

$\Rightarrow 4! = n(B) = \dots = n(E)$

$n(A\cap B) \Rightarrow$ Ball 1 in box 1 & Ball 2 in box 2

$= 3!$

$\Rightarrow n! - [{}^nC_1*(n-1)! - {}^nC_2*(n-2)! + {}^nC_3*(n-3)! + \dots + {}^nC_n(0)!$

$= \cancel{n!} - \cancel{n!} + {}^nC_2*(n-2)! - {}^nC_3*(n-3)! + \dots$ upto n terms

$= {}^nC_2*(n-2)! - {}^nC_3*(n-3)! + {}^nC_4*(n-4)! + \dots$ upto n

$= \frac{n!}{2!\cancel{(n-2)!}}\cancel{(n-2)!} - \frac{n!}{3!(n-3)!}(n-3)! + \frac{n!}{4!(n-4)!}(n-4)! + \dots$ upto n terms

$= \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} + \dots$

$$\boxed{D_n = \sum_{r=2}^{n} (-1)^r \frac{n!}{r!}}$$

Piegon Hole Principle-

- If n piegons occupy m piegon holes & n>m than atleast $\geq \left\lfloor\frac{n-1}{m}\right\rfloor+1$ piegons will ~~be~~ occupy same piegon holes.

Ex- 50 Bycycle painted with 7 colors. How many bicycle have same colors?

$m=7$
$n=50$

$\therefore$ No of ~~piegons~~ cycle $\geq \left\lfloor\frac{50-1}{7}\right\rfloor+1$

$\boxed{\text{No of } \cancel{\text{piegons}} \text{ cycle} \geq \cancel{8}\, 7+1 \geq 8}$

- If n piegons occupy m piegon holes & n>m than atleast $\left\lceil\frac{n}{m}\right\rceil$ piegons will occupy same piegon holes

- atmost - n

Que- Min No of cards to be dealt from a pack of 52 cards to guarantee that min 3 cards must be of same kind (suit)

$m=4$. $\left\lfloor\frac{n-1}{m}\right\rfloor+1 = 3$

$n-1 = 8$

$\boxed{n=9}$

• atleast 3 of same kind?

$m=13 \quad \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 3 \Rightarrow \boxed{n=27}$

• If 1 chit is made for every possible 3 men & 4 women committee obtained from 10 M & 15 W. these chit are dropped in 4 hats. 1 of these hats is guaranteed to have atleast how many chits?

$$n = {}^{10}C_3 * {}^{15}C_4 = \frac{10\times 9\times 8}{6} * \frac{15\times 14\times 13\times 12}{4\times 3\times 2} = 120 * 1365$$

$\therefore$ No of chits $\left\lfloor \frac{{}^{10}C_3 * {}^{15}C_4 - 1}{4} \right\rfloor + 1$

Ques- Consider a set of {1, 2, 3, 4, 5, ... 10} a pair of distinct no is selected again & again. Atleast how many of these pair have same sum?

$n = {}^{10}C_2 \quad$ min sum $=3 \quad$ max $=19 \quad \therefore$ No of sums $= 19-3+1 = 17$ (m)

$$\therefore = \left\lfloor \frac{{}^{10}C_2 - 1}{17} \right\rfloor + 1$$

Ques 10 pairs of R Socks, 12 pair of Green socks, 5 pair of Blue socks. All are mixed in bag. How many min. socks to be taken out to get 1 left & 1 right pair?

[28] $\quad$ or $\begin{bmatrix} R = 45 \\ G = 43 \\ B = 50 \end{bmatrix}$

Summation-

${}^nC_r$ - Binomial Coefficient

$(x+y)^n$ - ${}^nC_0 \; {}^nC_1 \ldots \ldots {}^nC_n \Rightarrow (n+1)$ terms

Ques- How many graphs are possible with n vertices?

Maximum No of edges $= \frac{n(n-1)}{2} \; ({}^nC_2)$

$\therefore$ No of graph possible

$$= {}^{n(n-1)/2}C_0 + {}^{\frac{n(n-1)}{2}}C_1 + {}^{\frac{n(n-1)}{2}}C_2 + \cdots + {}^{\frac{n(n-1)}{2}}C_{\frac{n(n-1)}{2}}$$

↑ zero edges graphs; ↑ 1 edge graphs; ↑ 2 edge graphs; ↑ $\frac{n(n-1)}{2}$ edged graphs

$$= 2^{\frac{n(n-1)}{2}}$$

SUMMATIONS:

- Symmetry: $^nC_r = {}^nC_{n-r}$
- Pascal's Identity: $\boxed{^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r}$

$^0C_0$

$^1C_0 \; {}^1C_1$

$^2C_0 \; {}^2C_1 \; {}^2C_2$

$^3C_0 \; {}^3C_1 \; {}^3C_2 \; {}^3C_3$

$^4C_0 \; {}^4C_1 \; {}^4C_2 \; {}^4C_3 \; {}^4C_4$

$\Rightarrow$

```
        1
       1 1
      1 2 1
     1 3 3 1
    1 4 6 4 1
```

$$\left[(x+y)^n = \sum {}^nC_r \, x^r y^{n-r}\right]$$

Proof-

$$^nC_r = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!} = \frac{(n-1)!}{(n-r-1)!\,(r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right]$$

$$= \frac{n(n-1)!}{(n-r)!\,r!} = \frac{n!}{r!(n-r)!} = {}^nC_r$$

$$\boxed{^nC_r = {}^{n-2}C_{r-2} + {}^{n-2}C_r + 2\cdot{}^{n-2}C_{r-1}}$$

$\uparrow$ x, y both included

$\uparrow$ x, y both excluded

$\uparrow$ x is included & y exclude or vice-versa

Newton's Identity: $\boxed{^nC_r * {}^rC_k = {}^nC_k * {}^{n-k}C_{r-k}}$

Ex- $^{100}C_{48} * {}^{48}C_{12} = {}^{100}C_{12} * {}^{88}C_{36} \equiv {}^{100}C_{88} * {}^{88}C_{36}$

Proof- Out of n people, r people are selected for a committee & k are choosen as leaders.

∴ We can do it like, select k leaders first from n & select non-leaders from the remaining people.

$$^nC_r * {}^rC_k = \underbrace{^nC_k}_{\text{selecting leaders}} * \underbrace{^{n-k}C_{r-k}}_{\text{Selecting non leaders}}$$

- used ~~for~~ when comb in multiplication.

Ques- On a 8×8 chess board, ways to place 8 white pawns & 8 Black pawns?

$$\boxed{{}^{64}C_{16} \times {}^{16}C_8 \times {}^8C_8} \equiv {}^{64}C_8 \times {}^{56}C_8$$
$$\equiv {}^{64}C_{56} \times {}^{56}C_8$$

4) **ROW SUM:**

$1 = 1 \quad (2^0)$

$1 \; 1 = 2 \quad (2^1)$

$1 \; 2 \; 1 = 4 \quad (2^2)$

$1 \; 3 \; 3 \; 1 = 8 \quad (2^3)$

$1 \; 4 \; 6 \; 4 \; 1 = 16 \quad (2^4)$

$$\therefore \boxed{{}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n}$$

$$\boxed{\sum_{r=0}^{n} {}^nC_r = 2^n}$$

- $\sum_{r=1}^{n} {}^nC_r = 2^n - 1$
- $\sum_{r=0}^{n-1} {}^nC_r = 2^n - 1$

Ques- How many subset of n elements have atleast one element?

$$\boxed{\text{No of subset possible} = 2^n - 1}$$

- atleast 2 element? $\boxed{\text{No of subset possible} = 2^n - n - 1}$

- $(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^r y^{n-r}$ put $x = 1, y = 1$

$$\boxed{2^n = \sum_{r=0}^{n} {}^nC_r}$$

Ques- $\sum_{r=0}^{10} {}^{10}C_r \times 2^r \, 3^{10-r} = ?$

$$\sum_{r=0}^{10} {}^{10}C_r \times 2^r \cdot 3^{10-r} = (x+y)^{10}$$
$$= (2+3)^{10} = 5^{10}$$

$$\sum_{r=0}^{10} \left(\tfrac{1}{2}\right)^{r} \left(\tfrac{3}{5}\right)^{n-r} {}^{10}C_r = \left(\tfrac{1}{2} + \tfrac{3}{5}\right)^{10} = \left(\tfrac{11}{10}\right)^{10}$$

$$\sum_{r=0}^{10} 2^{r}\, {}^{10}C_r \equiv \sum_{r=0}^{10} 2^{r}\, 1^{10-r}\, {}^{10}C_r \equiv (2+1)^{10} = 3^{10}$$

$$\sum_{r=0}^{10} 2^{-r}\, {}^{10}C_r \equiv \sum_{r=0}^{10} \left(\tfrac{1}{2}\right)^{r} (1)^{10-r}\, {}^{10}C_r \equiv \left(1+\tfrac{1}{2}\right)^{10}$$

∴ We can say

$$\boxed{\sum_{r=0}^{n} a^{r}\, {}^{n}C_r \equiv \sum_{r=0}^{n} a^{n-r}\, {}^{n}C_r \equiv (1+a)^{n}}$$

ques ${}^{10}C_0 + 2*{}^{10}C_1 + 4*{}^{10}C_2 + 8*{}^{10}C_3 + \dots\dots + 1024*{}^{10}C_{10} = ?$

${}^{10}C_1 * 2^0 \cdot 1^{10} + {}^{10}C_2 * 2^1 \cdot 1^9 + \dots\dots\ {}^{10}C_{10} * 2^{10}$

$\Rightarrow (2+1)^{10} = 3^{10}$

Q- How many subset of 10 element set include element 3?

1	2	3	4	5	6	7	8	9	10
2	2	2	1	2	2	2	2	2	2

ways

$= 2^9$ subset

• subset exclude element 3?

2^9 subset

• exclude 2 & include 7?

2^8 Subset

Q- 5 element subset which includes 2 & 7 always:

${}^{8}C_3$

• exclude 2, 7 always - ${}^{8}C_5$

• include 2 & exclude 7 - ${}^{8}C_4$

5. Alternating Sign Row Sum:

$$^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - \cdots {}^nC_n = 0$$

$$\boxed{\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0}$$

- $\sum_{K=0}^{10} (-1)^K \, {}^{10}C_K = 0$
- $\sum_{K=1}^{10} (-1)^K \, {}^{10}C_K = \sum_{K=0}^{10} (-1)^K \, {}^{10}C_K - {}^{10}C_0 = -1$

proof- $\sum_{r=0}^{n} (-1)^r \, {}^nC_r = (1+a)^n$

where $a=-1$

$= (1-1)^n = 0$

- $\sum_{K=0}^{n} (-3)^K \, {}^nC_K = (-2)^n$

6. Even Sum & odd Sum:

$$\left[{}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 = 2^{n-1}\right]$$

$$\boxed{\sum_{K=0}^{n/2} {}^nC_{2K} = \sum_{K=1}^{n/2} {}^nC_{2K-1} = 2^{n-1}}$$

Ques- How many subset for n element with even no of elements

$$\left[{}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + \cdots = 2^{n-1}\right]$$

Ques $\frac{1!}{0!\,10!} + \frac{1}{2!\,8!} + \frac{1}{4!\,6!} + \frac{1}{6!\,4!} + \cdots \frac{1}{10!\,0!} = \frac{2^m}{n!}$

$$X = \frac{1}{0!\,10!} + \frac{1}{2!\,8!} + \cdots \frac{1}{10!\,0!} = \frac{\left[{}^{10}C_0 + {}^{10}C_2 + {}^{10}C_4 + \cdots {}^{10}C_{10}\right]}{10!}$$

$$= \frac{2^{10-1}}{10!} = \frac{2^9}{10!}$$

$$\boxed{m = 9,\ n = 10}$$

Row Square Summation

$$^nC_0^2 + ^nC_1^2 + ^nC_2^2 + \dots + ^nC_n^2 = ^{2n}C_n$$

$$\boxed{\sum_{r=0}^{n} {}^nC_r^2 = {}^{2n}C_n}$$

> Column Summation:

$$^rC_r + ^{r+1}C_r + ^{r+2}C_r + \dots + ^nC_r = ^{n+1}C_{r+1}$$

$$\boxed{\sum_{k=r}^{n} {}^kC_r = {}^{n+1}C_{r+1}}$$

Ex- $^5C_5 + ^6C_5 + \dots \dots ^9C_5 = ^{10}C_6$

$^5C_5 + ^6C_5 + \dots \dots ^{100}C_5 = ^{101}C_6$

$^{60}C_5 + ^{61}C_5 + \dots \dots ^{100}C_5 = ^{101}C_6 - ^{60}C_6$

$^0C_5 + ^1C_5 + \dots \dots ^9C_5 = ^{10}C_6$

$$\boxed{\sum_{K=r}^{n} {}^KC_r = {}^{n+1}C_{r+1} \equiv \sum_{K=0}^{n} {}^KC_r = {}^{n+1}C_{r+1}}$$

Note- $^nC_r = 0$ when $n < r$

$$\left[\sum_{\substack{K=x \\ x>r}}^{n} {}^KC_r \equiv \sum_{K=0}^{n} {}^KC_r - \sum_{K=0}^{x-1} {}^KC_r = {}^{n+1}C_{r+1} - {}^xC_{r+1}\right]$$

9. VanderMonde's Identity

To select r item from $m+n$ element, where m of one kind & n of other kind,

then,

$$\boxed{{}^{m}C_0\,{}^{n}C_r + {}^{m}C_1\,{}^{n}C_{r-1} + {}^{m}C_2\,{}^{n}C_{r-2} + \dots {}^{m}C_r \cdot {}^{n}C_0 = {}^{m+n}C_r}$$

$$\boxed{\sum_{k=0}^{r} {}^{m}C_k\,{}^{n}C_{r-k} = {}^{m+n}C_r}$$

Ex- Out of 25 people with 10 B & 15 G selecting 5 people as

25

10 B | 15 G

5

0	5
1	4
2	3
3	2
4	1
5	0

$$\Rightarrow \quad {}^{25}C_5 = {}^{10}C_0 * {}^{15}C_5 + {}^{10}C_1 * {}^{15}C_4 + {}^{10}C_2 * {}^{15}C_3 + {}^{10}C_3 * {}^{15}C_2 + {}^{10}C_4 * {}^{15}C_1 + {}^{10}C_5 * {}^{15}C_0$$

Special case 1. when $m = n = r = n$

$${}^{n}C_0 \cdot {}^{n}C_n + {}^{n}C_1 \cdot {}^{n}C_{n-1} + {}^{n}C_2 \cdot {}^{n}C_{n-2} + \dots + {}^{n}C_n \cdot {}^{n}C_0 = {}^{n+n}C_n$$

$$\boxed{{}^{n}C_0^{\,2} + {}^{n}C_1^{\,2} + {}^{n}C_2^{\,2} + \dots {}^{n}C_n^{\,2} = {}^{2n}C_n}$$

$$\sum_{r=0}^{12} {}^{10}C_r * {}^{15}C_{12-r} = {}^{25}C_{12}$$

proof for Row Square sum

10) $$\sum_{r=0}^{n} r \cdot {}^{n}C_r = n \cdot 2^{n-1}$$

$$0 \cdot {}^{10}C_0 + 1 \cdot {}^{10}C_1 + 2 \cdot {}^{10}C_2 + \dots + 10 \cdot {}^{10}C_{10} = 10 \cdot 2^9$$

No of terms in binomial expression with $(x+y)^n = n+1$

Coefficient of $x^r y^{n-r}$ = coefficient of $x^{n-r} y^r$.

Coefficient of $x^3 y^9$ in $(x+y)^{10} = 0$

What is the coefficient of x^{12} in $\left(x^2 + \frac{1}{x^3}\right)^{10}$?

$$\left(x^2 + \frac{1}{x^3}\right)^{10} = \sum_{r=0}^{10} {}^{10}C_r \, (x^2)^r \left(\frac{1}{x^3}\right)^{10-r}$$

$$= \sum_{r=0}^{10} {}^{10}C_r \, x^{2r} \, x^{-3(10-r)}$$

$$= \sum_{r=0}^{10} {}^{10}C_r \, x^{2r-30+3r}$$

$$= \sum_{r=0}^{10} {}^{10}C_r \, x^{5r-30}$$

$5r - 30 = 12$

$r = 42/5 \rightarrow$ not an integer

Coefficient of $x^{12} = 0$

Coefficient of x^{15} = $5r - 30 = 15$

$r = 9$

$\therefore \boxed{\text{Coefficient} = {}^{10}C_9 = 10}$

Ques When $(x+y)(x+y) \ldots$ 10 times than what is the no of terms with x^3y^7?

No of terms = coefficient of x^3y^7 in $(x+y)^{10}$

$= {}^{10}C_3 = 120$

Note- $(x+y)^n = \sum_{r=0}^{n} {}^nC_r = 2^n$

$= \frac{n!}{r!(n-r)!} = 2^n$ $\quad r = n_1 \quad n - r = n_2 \quad n_1 + n_2 = n$

for 3 variables

$$\left[\sum_{n_1+n_2+n_3=n}^{n} \frac{n!}{n_1!\,n_2!\,n_3!} = 3^n\right]$$

$$\boxed{(x+y+z)^n = \sum_{n_1+n_2+n_3=n} \frac{n!}{n_1!\,n_2!\,n_3!}\, x^{n_1} y^{n_2} z^{n_3}}$$

[Multi Nomial Expansion]

- if you expand $(x+y+z)^{10}$, what is the coffiαient of $x^2y^3z^5$?

$$\frac{10!}{2!\,3!\,5!}$$

- coefficient of $x^3y^3z^5 = 0$
- coefficient of $x^{10}y^0z^0 = \frac{10!}{10!\,0!\,0!}$

- No of terms in expansion of $(x+y+z)^{10}$

$x_1 + x_2 + x_3 = 10 \Rightarrow {}^{12}C_{10} \Rightarrow \frac{12\times 11}{2} = 66$ terms

$\left[{}^{n-1+r}C_r\right]$

Generating Functions

[calculus of discrete function]

Sequence	Numeric function $[a_r]_0^\infty$	Generating f^n $\sum_{r=0}^{\infty} a_r x^r$
1, 1, 1, ...	1	$A(x) = 1+x+x^2+x^3+\dots = \frac{1}{1-x}$ $\left[\sum_{r=0}^{\infty} 1 \cdot x^r = 1/_{1-x}\right]$
1, -1, 1, -1, +1, -1, ...	$(-1)^r$	$\sum_{r=0}^{\infty} (-1)^r x^r = \frac{1}{1+x}$
1, a, a^2, a^3, a^4, ...	a^r	$\sum_{r=0}^{\infty} a^r x^r = \frac{1}{1-ax}$
1, -a, a^2, $-a^3$, ...	$(-a)^r = (-1)^r a^r$	$\sum_{r=0}^{\infty} (-a)^r x^r = \frac{1}{1+ax}$
1, 0, 1, 0, 1, 0, ...	$a_r = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 1 & \text{if } r \text{ is even} \end{cases}$	$\sum_{r=0}^{\infty} (x^2)^r = \frac{1}{1-x^2}$ $1+x^2+x^4+x^6+\dots$
1, 0, 0, 1, ~~0, 0~~, 1, 0, 0	$a_r = \begin{cases} 0 & r \neq 3x \\ 1 & r = 3x \end{cases}$	$\sum_{r=0}^{\infty} (x^3)^r = \frac{1}{1-x^3}$
	${}^{n-1+r}C_r$	$\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} \left({}^{n-1+r}C_r \cdot x^r\right)$
	${}^{n}C_r$	$(1+x)^n$
$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots$	$\frac{1}{r!}$	$\sum_{r=0}^{\infty} \frac{1}{r!} \cdot x^r = e^x$

Properties-

1. if $A(x)$ corresponds to $[a_r]$ & $B(x)$ corresponds to b_r than

(a) $\boxed{a_r \pm b_r = A(x) \pm B(x)}$

(b) $K_1 \cdot a_r \pm K_2 \cdot b_r = K_1 \cdot A(x) \pm K_2 \cdot B(x)$ [Linearity Property]

K_1, K_2 - constant

Ques what is the Generating f^n corresponding to

$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

$$\left[\sum_{r=0}^{\infty} \left(\frac{1}{3}\right)^r x^r = \frac{3}{3-x}\right]$$

- what is the coefficient of x^{21}?

$$\frac{3}{3-x} = \frac{1}{1-(\frac{1}{3})x} \Rightarrow a = \frac{1}{3}$$

$$\therefore \text{coefficient} = \left(\frac{1}{3}\right)^{21}$$

2. If $A(x)$ corresponds to a_r than

$$K \cdot a_r \Rightarrow K \cdot A(x)$$

Ques-

$$\sum_{r=0}^{\infty}\left(2^r - \frac{1}{3^r}\right)x^r = \sum_{r=0}^{\infty} 2^r x^r - \sum_{r=0}^{\infty} \frac{1}{3^r} x^r$$

$$= \frac{1}{1-2x} - \frac{1}{1-\frac{1}{3}x}$$

$$= \frac{1}{1-2x} - \frac{3}{3-x} = \frac{3-x-3+6x}{(1-2x)(3-x)} = \frac{5x}{(1-2x)(3-x)}$$

$$\text{Coefficient of } x^{10} = \left(2^{10} - \frac{1}{3^{10}}\right) = \frac{2^{10} \cdot 3^{10} - 1}{3^{10}}$$

3:> if a_r corresponds to Ax

then $A(ax) = \sum_{r=0}^{\infty} a_r (a x)^r = \sum_{r=0}^{\infty} a_r\, a^r x^r$

Ques- What is the coefficient of x^7 in $(1+3x)^{10}$

$$\left[{}^{10}C_7 * (3)^7\right]$$

$$\downarrow \quad - \quad \downarrow a^r$$

$$\therefore a_r$$

ues $\sum_{r=0}^{\infty} (-1)^r \, 3^r \, {}^{10}C_r \, x^r$?

$G(z) = (1-3x)^{10}$

at is a_r corresponding to $(1-5x)^{15}$

$$\left[a_r = {}^{15}C_r \cdot 5^r \, (-1)^r\right]$$

ues- $\sum_{r=0}^{\infty} {}^{10+r}C_r \, x^r = \sum_{r=0}^{\infty} {}^{11-1+r}C_r \, x^r = \frac{1}{(1-x)^{11}}$

ues- $\frac{1}{(1+x)^{15}} = \sum_{r=0}^{\infty} {}^{15-1+r}C_r \, (-1)^r - x^r \qquad \frac{1}{1+x} = \frac{1}{1-(-x)}$

ues $\frac{1}{1-5x^3} = \sum_{r=0}^{\infty} x^{3r} \cdot 5^r$

es- In $\frac{1}{(1+3x)^{11}}$, what is coefficient of x^{12}?

$$\frac{1}{(1+3x)^{11}} = \sum_{r=0}^{\infty} {}^{11-1+r}C_r \; 3^r \cdot (-1)^r \cdot x^r$$

$$= \sum_{r=0}^{\infty} {}^{10+r}C_r \; 3^r \cdot (-1)^r \cdot x^r$$

for $x^{12} \Rightarrow$ coefficient $= {}^{22}C_{12} \cdot 3^{12}$

ote- $\left[\frac{1}{(1+x)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r \equiv \sum_{r=0}^{n} {}^{n+r-1}C_r \, x^r\right]$

Ques- $\sum_{r=0}^{\infty} (-1)^r \cdot 3^r \cdot \frac{5 \times x^r}{r!} = e^{-3x} \cdot 5 = 5e^{-3x}$

Table II-

Sequence	Numeric function	Generating function
1, 1, 1, 1, ...	{1}	$\frac{1}{1-x}$
0, 1, 2, 3, 4, 5	r	$\frac{x}{(1-x)^2}$
0, 1, 4, 9, 16, ...	r^2	$\frac{x(1+x)}{(1-x)^3}$

$$\sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r = \frac{1}{(1-x)^n}$$

Let $n=1$

$$\sum_{r=0}^{\infty} {}^{\cancel{r}\,1}C_r \cdot x^r = \frac{1}{1-x} \Rightarrow \sum_{r=0}^{\infty} 1 \cdot x^r = \frac{1}{1-x}$$

Let $n=2$

$$\sum_{r=0}^{\infty} {}^{1+r}C_r \cdot x^r = \frac{1}{(1-x)^2}$$

$$\Rightarrow \sum_{r=0}^{\infty} (r+1) x^r = \frac{1}{(1-x)^2} \Rightarrow \sum_{r=0}^{\infty} r x^r = \frac{1}{(1-x)^2} - \sum_{r=0}^{\infty} x^r$$

$$= \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{1-1+x}{(1-x)^2}$$

$$= \frac{x}{(1-x)^2}$$

Let $n=3$ $\sum_{r=0}^{\infty} \frac{(r+2)(r+1)}{2} \cdot x^r = \frac{1}{(1-x)^3}$

$$\sum_{r=0}^{\infty} r^2 x^r = \frac{2}{(1-x)^3} - \frac{3x}{(1-x)^2} - \frac{2}{1-x} = \frac{\cancel{2} - 3x + 3x^2 - \cancel{2} - 2x^2 + 4x}{(1-x)^3} = \frac{x^2+x}{1-x^3}$$

Ques $\sum_{r=0}^{\infty} (r^2+3r+2)x^r$

$$= \frac{x(1+x)}{(1-x)^3} + \frac{3x}{(1-x)^2} + \frac{2}{(1-x)} = \frac{\cancel{x}+\cancel{x^2}+\cancel{3x}-\cancel{3x^2}+2+\cancel{2x^2}-\cancel{4x}}{(1-x)^3} = \frac{2}{(1-x)^3}$$

Ques $\sum_{i=0}^{\infty} (i+1)z^i = ?$

$$= \frac{z}{(1-z)^2} + \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

Ques $\frac{z^2}{(1-z)^3}$ What is $a_3 - 5a_2$?

$$\Rightarrow \cancel{\frac{z^2+z}{(1-z)^3}} - \cancel{\frac{z}{(1-z)^3}}$$

$$\cancel{\sum_{r=0}^{\infty} r^2 z^r} -$$

$$\frac{1}{(1-z)^3} = \left[{}^{n+r-1}C_r \Rightarrow a_r \right]$$

$1, {}^nC_1, {}^{n+1}C_2, {}^{n+2}C_3 \longrightarrow \frac{1}{(1-z)^3}$

$0, 1, {}^nC_1, {}^{n+1}C_2, {}^{n+2}C_3 \longrightarrow \frac{z}{(1-z)^3}$

$0, 0, 1, {}^nC_1, {}^{n+1}C_2, {}^{n+2}C_3 \longrightarrow \frac{z^2}{(1-z)^3}$

$\therefore a_3 = {}^nC_1 \quad a_2 = 1 \Rightarrow a_3 - 5a_2 = n-5$

Property 5:- Shifting property of generating function:

If the sequence $a_0, a_1, a_2, a_3 \ldots a_n$ corresponds to $A(x)$

then, $0, a_0, a_1, a_2 \ldots \longrightarrow x\,A(x)$

ie if $\left[\begin{array}{l} a^r \text{ was coefficient of } x^r \\ a^r \text{ ——— } x^{r+1} \end{array} \right]$

Ques What G.F^n 0, 0, 1, 1, 1, 1 corresponds to?

$1, 1, 1, 1, 1 \ldots$ $\quad Gf = \frac{1}{1-x}$

$0, 1, 1, 1, 1 \longrightarrow \frac{1}{1-x} \cdot x$

$0, 0, 1, 1, 1, 1 \longrightarrow \frac{x^2}{1-x}$

Ques- $\frac{1+z}{(1-z)^3}$ what is $a_3 - a_0$

$\frac{1}{(1-z)^3} + \frac{z}{(1-z)^3}$

$\downarrow$

$a_n = {}^{n+r-1}C_r$

$\Downarrow$

$a_3 \Rightarrow {}^{n+2}C_3$

$a_0 \Rightarrow 1$

$\frac{1}{(1-z)^3} \Rightarrow a_r = {}^{n+r-1}C_r$

	a_0	a_1	a_2	a_3	a_4
$\frac{1}{(1-z)^3}$	1	n	${}^{n+1}C_2$	${}^{n+2}C_3$	${}^{n+3}C_4$
$\frac{z}{(1-z)^3}$	0	1	n	${}^{n+1}C_2$	${}^{n+2}C_3$

[using shifting property]

$a_0 = 1$

$a_3 = {}^{n+1}C_2$

$\therefore$ ~~$a_r = {}^{n+r-1}C_r$~~ +

$\therefore a_3 - a_0 = {}^{n+2}C_3 + {}^{n+1}C_2 - 1 - 1$

Here $n = 3$

$= {}^5C_3 + {}^4C_2 - 2 = \frac{5*4}{2} + \frac{4*3}{2} - 2 = 10 + 6 - 2 = \underline{\underline{14}}$

what is $x^2 e^x = \sum$

$e^x = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} \frac{x^r}{r!} \Rightarrow a_r = \frac{1}{r!}$

$x^2 e^x \Rightarrow a_r = \frac{1}{(r-2)!}$ [It will perform the direct shifting as]

a_0	a_1	a_2	a_3
0	0	0	1

$\underbrace{0 \quad 0}_{2\text{ shift}}$

to find $a_{10} - 5a_7$

$\Downarrow$

$a_8 - 5a_5$

$= \left[\frac{1}{8!} - \frac{5}{5!}\right]$

$\therefore x^2 e^x = \sum_{r=0}^{\infty} a_r x^r$ where $a_r = \begin{cases} \frac{1}{(r-2)!} & r \geq 2 \\ 0 & r < 2 \end{cases}$

Ques what is the coefficient of a_r in $\frac{x^4(1+x)}{(1-x)^3}$?

$$\frac{x^4(1+x)}{(1-x)^3} = x^3 \cdot \left[\frac{x(1+x)}{(1-x)^3}\right]$$

$$= x^3 \cdot \sum_{r=0}^{\infty} a_r \cdot x^r \quad a_r = r^2$$

to balance x^3, shifting of value of r is needed from 3 places.

$$a_r = \begin{cases} (r-3)^2 & r \geq 3 \\ 0 & r < 3 \end{cases}$$

$$\frac{z^2}{(1-z)^3} \quad \text{for } \frac{1}{(1-z)^3} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r = \sum_{r=0}^{\infty} {}^{2+r}C_r \cdot x^r$$

for again shifting by 2.

$$r = r-2$$

$$a_r = {}^{2+r-2}C_{r-2} \cdot x^r$$

$$= {}^{r}C_{r-2} \cdot x^r$$

$$\boxed{a_r = {}^{r}C_2}$$

Ques- 1. $\frac{1}{(3z+2)} = \frac{1}{2+3z} = \frac{1/2}{1+3/2 z}$

$$\therefore a_r = 1/2 * (-3/2)^r = \frac{1}{2} (-1)^r (3/2)^r \text{ Ans}$$

2. $\frac{1}{2-3z} = \frac{1/2}{1-3/2 z} \Rightarrow a_r = 1/2 * (3/2)^r$

3. $\frac{1}{3z-2} = \left(\frac{-1}{2}\right) \frac{1}{1-3/2 z} \Rightarrow a_r = (-1/2) * (3/2)^r$

4. $\frac{1}{z^2-5z+6} = \frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2} \Rightarrow \cancel{a_r = (3)^r + 2^r}$

$\cancel{= 2^r - 3^r}$

[using partial fraction Method

Partial Fraction Method

$$\frac{1}{(z-2)(z-3)} = \frac{A}{(z-2)} + \frac{B}{(z-3)}$$

$$A(z-3) + B(z-2) = 1$$

$$A = -1$$

$$B = 1$$

$$\frac{1}{z-3} - \frac{1}{z-2} = \left(\frac{-1}{3}\right)\frac{1}{1-\frac{1}{3}z} + \frac{1}{2}\frac{1}{1-\frac{1}{2}z}$$

$$\therefore a_n = \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n$$

$$= -3^{n+1} + 2^{n+1}$$

$$\boxed{a_n = 2^{n+1} - 3^{n+1}}$$

5. $\frac{z^3}{z^2-5z+6}$ $\Rightarrow$ put $n = n-3$

$$a_n = 2^{n-3+1} - 3^{n-3+1}$$

$$= 2^{n-2} - 3^{n-2}$$

$$\begin{bmatrix} a_n = 2^{n-2} - 3^{n-2} & n \geq 2 \\ = 0 & n < 2 \end{bmatrix}$$

Solving upper constraint problem:

1) DICE SUM PROBLEM

$$x_1 + x_2 + x_3 = 12$$

$$1 \leq x_1, x_2, x_3 \leq 6$$

- Allow powers only upto the given limit

$$A(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + \cdots + x^6)(x^1 + x^2 + \ldots + x^6)$$

Now search for coefficient of x^{12},

$$= (x + x^2 + x^3 + x^4 + x^5 + x^6)^3$$

$$= x^3(1 + x + x^2 + x^3 + x^4 + x^5)^3$$

$$= x^3\left(\frac{1\cdot(x^6-1)}{(x-1)}\right)^3$$

$$= \frac{x^3 \cdot (x^6-1)^3}{(x-1)^3}$$

- coefficient of x^{12} tells about all the combination 3 dice can result into sum=7.

$$\Rightarrow \frac{x^3(1-x^6)^3}{(1-x)^3}$$

$$\Rightarrow x^3(1-x^6)^3\left[\sum_{n=0}^{\infty} {}^{3+n-1}C_n x^n\right]$$

$$\Rightarrow x^3(1-x^6)^3\left[\sum_{n=0}^{\infty} {}^{2+n}C_n x^n\right]$$

$$\Rightarrow x^3\left[{}^3C_0 + {}^3C_1(-x^6) + {}^3C_2(-x^6)^2 + {}^3C_3(-x^6)^3\right] \sum_{n=0}^{\infty} {}^{n+2}C_n x^n$$

$$\Rightarrow \left[x^3 - 3x^9 + \cancel{3x^{15}} - \cancel{x^{21}}\right] \sum_{n=0}^{\infty} {}^{n+2}C_n x^n$$

[Cannever lead to x^{12}]

$$\Rightarrow \left[x^3 - 3x^9\right] \sum_{n=0}^{\infty} {}^{n+2}C_n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} {}^{n+2}C_n x^{n+3} - 3\sum_{n=0}^{\infty} {}^{n+2}C_n x^{n+9}$$

Let $n = 9$ Let $n = 3$

$$= {}^{11}C_9 - {}^5C_3 \times 3$$

$$= \frac{11\times10}{2} - \frac{5\times 4\times 3}{2} = 55 - 30 = 25$$

Digit Sum problem

$$x_1 + x_2 + x_3 \leq 12$$

$$0 \leq x_1, x_2, x_3 \leq 9$$

$$A(x) = (1 + x + x^2 + x^3 + \dots + x^9)^3$$

$$= 1\cdot\left[\frac{(1-x^{10})}{1-x}\right]^3$$

$$= (1-x^{10})^3 \sum_{n=0}^{\infty} {}^{2+n}C_n x^n$$

for x^{12}

$$= (1 - \cancel{x^{30}} - 3x^{10} + \cancel{3x^{20}}) \sum_{n=0}^{\infty} {}^{n+2}C_n x^n$$

$$= \sum_{n=0}^{\infty} {}^{n+2}C_2 x^n - 3\sum_{n=0}^{\infty} {}^{n+2}C_n x^{n+10}$$

$$= {}^{14}C_2 - 3\times{}^4C_2 \Rightarrow \frac{14\times13}{2} - \frac{3\times 4\times3}{2} = 91 - 18 = 73$$

Ques- Ways to put $3n$ identical balls into 2 boxes so that each box contain atmost $2n$ ball?

$$x_1 + x_2 = 3n$$
$$0 \leq x_1, x_2 \leq 2n$$

$$A(x) = (1 + x + x^2 + x^3 + \dots + x^{2n})^2$$
$$= \left[\frac{1 \cdot (1 - x^{2n+1})}{(1-x)}\right]^2$$
$$= (1 - x^{2n+1})^2 \sum_{r=0}^{\infty} {}^{r+1}C_r\, x^r$$
$$= \left(1 + \cancel{x^{4n+2}} - 2 \cdot x^{2n+1}\right) \sum_{r=0}^{\infty} (r+1)\, x^r$$
$$= \sum_{r=0}^{\infty} (r+1) x^r - 2 \sum_{r=0}^{\infty} (r+1)\, x^{2n+1+r}$$
$$r = 3n \qquad\qquad r = n-1$$
$$= 3n+1 - 2(n - \cancel{1} + \cancel{1})$$
$$= 3n + 1 - 2n = \underline{n+1}$$

Ques- 3 jars 1 with 5 R 2^{nd} with 25 G 3^{rd} with 8 B choclate

Way to select atleast 2 R & atleast 1 Green choclate the selection of 20 choclate is done.

$$x_1 + x_2 + x_3 = 20$$
$$2 \leq x_1 \leq 5 \qquad x_2 \geq 1$$
$$0 \leq x_3 \leq 8$$

$$\boxed{\begin{array}{l} 0 \leq x_1 \leq 3 \quad x_2 \geq 0 \\ 0 \leq x_3 \leq 8 \\ x_1 + x_2 + x_3 = 17 \end{array}}$$

$$\Rightarrow (x^2 + x^3 + x^4 + x^5)(x + x^2 + x^3 + \dots + \cancel{x^{20}})(\cancel{1} + x + \dots + x^8)$$
$$\cancel{x^3}(1 + x + x^2 + x^3)(1 + x^2 + x^3 + \dots \cancel{x^{25}})(1 + x + \dots + x^8)$$

↳ No upper const

[even if you put it, it will result incorrect answer]

$$\frac{1(1-x^4)}{(1-x)} \cdot \frac{(1-\cancel{x^{20}})}{(1-x)} \cdot \frac{(1-x^9)}{(1-x)}$$
$$= (1 - x^4)(1 - \cancel{x^{20}})(1 - x^9) \sum_{r=0}^{\infty} {}^{2+r}C_2\, x^r$$
$$= {}^{19}C_2 - {}^{10}C_2 - {}^{15}C_2 + {}^{6}C_2 = \cancel{19 \times 9 - 5 \times 9 - 9 \times 17 + 9 \times 4}$$
$$= \cancel{18 + 36 - 45}$$
$$= 19 \times 9 - 45 - 105 + 15 = 186 - 150 = 36$$

Solution of Reccurence Relation

Ques- Recc. Rel^n for no of binary string not containing 11.

for 0 length = 1
1 length = 2
2 length = 3
3 length = 5

$\boxed{a_n = a_{n-1} + a_{n-2}}$

Relations like $[a_n = 3a_{n-1}]$ are reccurence relations.

Reccurence Relation
- Determinate order
  - Linear (an term should be Linear)
    - LRCC (Linear Rec. Rel^n with const. coefficient)
      - [LRCCH] (RHS=0) Homogeneous
      - (RHS ≠ 0) Inhomogeneous
    - LRVC (with variable coefficient)
  - [Substit^n]
  - Non Linear $[a_n^2 = 5a_{n-1} + 6]$
  - [by substitution]
- Indeterminate order

Inhomogeneous
- polynomial
- power
- power with polynomial

Determinate Order

$a_n = a_{n-1} + 6a_{n-2}$ $\quad$ [Order = $n-(n-2) = \boxed{2}$]

[Linear with order 2]

Indeterminate order

$a_n = a_{n/2} + 6a_{n-5}$ $\quad$ [order = $n - n/2 = n/2$]

↳ variable order

$a_n = a_{n/d} + f(n)$

↳ ~~inderm~~ indeterminate equ^n

$a_n = a_{n-1} a_{n-2}$ → Non Linear

$a_n = \sqrt{a_{n-1}} + 6$

- sin, cosine etc are also not allowed.

$a_n = 5a_{n-1} + 6a_{n-2}$

$a_n = a_{n-1} + n^2$ → LRCC

$a_n = n\,a_{n-1}$ → LRVC

Homogeneous- RHS = 0

$a_n - 7a_{n-1} - 5a_{n-2} = 0$

Inhomogeneous

$a_n - 7a_{n-1} = 5$

$a_n - 7a_{n-1} = n * 2^n$

$a_n - 7a_{n-1} = n$

$a_n - 7a_{n-1} = 2^n$

Solving of Homogeneous Recc. Relation :

$a_n = 5a_{n-1} - 6a_{n-2}$

$a_n - 5a_{n-1} + 6a_{n-2} = 0$ $[a_0 = 5 \quad a_1 = 3]$

Characteristic Root Method :

To convert into polynomial equ. put lower index to 1 (t^0) & so on.

$t^2 - 5t + 6 = 0$

$t^2 - 3t - 2t + 6 = 0$

$(t-3)(t-2) = 0$

$\boxed{t = 3, 2}$

Note- • If α, β are roots for equ^n

$\boxed{A_n = C_1\alpha^n + C_2\beta^n}$

$= C_1 2^n + C_2 3^n$

$5 = C_1 + C_2$

$3 = 2C_1 + 3C_2 \Rightarrow \boxed{a_n = 12 \cdot 2^n - 7 \cdot 3^n}$

$15 = 3C_1 + 3C_2$

$C_1 = 12 \quad C_2 = -7$

• If one of root is 1.

$|A(n) = C_1 1^n + C_2\beta^n$

$= C_1 + C_2\beta^n$

• If root are complex No, do it directly

$\alpha_1 = 3+i \quad \beta_2 = \beta\, 3-i$

$a_n = C_1(3+i)^n + C_2(3-i)^n$

• for repeated roots $\alpha = \beta$

Ex $t^2 - 4t + 4 \quad t = 2, 2$

$\boxed{a_n = C_1 2^n + n \cdot C_2 2^n}$

• for 3 ~~real~~ roots $(2, 3, 5)$

$a_n = 2^n + C_2 3^n + C_3 5^n$

If roots are 2, 2, 3

$$a_n = c_1 2^n + c_2 n 2^n + c_3 3^n$$

If roots are 2, 2, 2

$$\boxed{a_n = c_1 \cdot 2^n + c_2 \cdot n 2^n + c_3 \cdot n^2 2^n}$$

- for solving $t^3 - 5t^2 + 6t + 7 = 0$ (Solving cubic equ^n)
  - try to use options.

inHomogeneous :
with constant

$$a_n = 5a_{n-1} - 6a_{n-2} + 3 \qquad a_0 = 5 \quad a_1 = 3$$

$$\boxed{a_n - 5a_{n-1} + 6a_{n-2} = 3}$$

↑ constant f^n

Solution-

$$a_n = \underbrace{a_n^h}_{\text{homogeneous Sol}^n} + \underbrace{a_n^p}_{\text{particular Sol}^n}$$

Homogeneous Sol^n :

$$a_n^h \Rightarrow a_n - 5a_{n-1} + 6a_{n-2} = 0$$

$$t^2 - 5t + 6 = 0$$

$$\boxed{t = 2, 3}$$

$$\therefore a_n^h = c_1 \cdot 2^n + c_2 \cdot 3^n$$

Particular Sol^n

	R.H.S of equ	particular sol^n
	c	d
polynomial	$c_0 + c_1 n$	$d_0 + d_1 n$
	$c_0 + c_1 n + c_2 n^2$	$d_0 + d_1 n + d_2 n^2$
power	$c \cdot a^n$	$d \cdot a^n$
polynomial with power	$(c_0 + c_1 n) a^n$	$(d_0 + d_1 n) a^n$

- particular sol^n satisfy the equ^n

$$a_n^p = d$$

$$\therefore a_n = d$$

$$d - 5d + 6d = 3$$
$$2d = 3 \qquad \Rightarrow d = 3/2$$

$$a_n = c_1 2^n + c_2 3^n + d$$
$$= c_1 2^n + c_2 3^n + 3/2$$

$a_0 \Rightarrow c_1 + c_2 + 3/2 = 5$

$a_1 \Rightarrow 2c_1 + 3c_2 + 3/2 = 8$

$c_1 + c_2 = 7/2 \quad = \quad 3c_1 + 3c_2 = 21/2 \qquad \Rightarrow c_1 = 4$

$2c_1 + 3c_2 = 13/2 \qquad 2c_1 + 3c_2 = 13/2 \qquad \therefore c_2 = -1/2$

$$\therefore \boxed{a_n = 4\cdot 2^n + 3/2 - 1/2\, 3^n}$$

Note- There should be no collapsing b/w terms of C & d $[a^p_n \ \& \ a^h_n]$.

Ex- $t = 1, 1 \quad a^p_n = d$

$\therefore a^h_n = c_1 + c_2 n$

$a^p_n = d$

$\boxed{\text{make changes before obtaining value of d.}}$

$a_n = \underbrace{c_1 + c_2 n + d}_{\text{collision}} \Rightarrow \underbrace{c_1 + c_2 n + dn}_{\text{again collision}} \Rightarrow c_1 + c_2 n + dn^2$

↑ No collision

Ques- $a_n - 5a_{n-1} + 6a_{n-2} = 3n$

$a_n = a^h_n + a^p_n$

$a^h_n = c_1 2^n + c_2 3^n$

$a^p_n = d_0 + d_1 n$

$d_0 + d_1 n - 5(d_0 + d_1(n-1)) + 6(d_0 + d_1(n-2)) = 3n$

$d_0 - 5d_0 + 6d_0 + 5d_1 - 12d_1 + (d_1 n - 5d_1 n + 6d_1 n) = 3n$

by comp, $(2d_0 - 7d_1) = 0 \Rightarrow d_0 = \frac{7}{2} d_1$

$2d_1 = 3 \Rightarrow d_1 = 3/2$

$\therefore d_0 = \frac{21}{4}$

$a^p_n = \frac{21}{4} + \frac{3}{2} n$

$$\therefore \boxed{a_n = c_1 2^n + c_2 3^n + \frac{21}{4} + \frac{3}{2} n}$$

Ques $a_n = a_{n-1} + n^2 + n$ $a_1 = 1$ what is a_{99}?

$a_2 = a_1 + 2^2 + 2$

$a_3 = a_2 + 3^2 + 3$

$= a_1 + 2^2 + 2 + 3^2 + 3$

$a_{99} = a_1 + 99^2 + \dots + 2^2 + 2 + 3 + \dots 99$

$a_{100} \equiv a_1 + 2^2 + 3^2 + \dots + 100^2 + 2 + 3 + \dots 100$

$= a_1 + \left[\frac{100 * 10\cancel{0} * 201}{6} - 1\right] + \left[\frac{100 * 10\cancel{0}}{2} - 1\right]$

$= \cancel{1} + \left[\frac{100 * 101 * 201}{6}\right] \cancel{-1} + \left[\frac{100 * 101}{2}\right] - 1$

$= \frac{100 * 101}{2}\left[\frac{201}{3} - 1\right] - 1$

$= \frac{100 * 101 \; \cancel{*198} \; 66 \; 33}{\cancel{2} * \cancel{3}} - 1$

$= 10100 * 33 - 1$

$= 3303300 - 1$

$= \underline{3303299}$

by using substitution Method-

1. $\sum n = \frac{n(n+1)}{2}$
2. $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum n^3 = \left[\left(\frac{n(n+1)}{2}\right)^2\right]$

Ques- $\left[\begin{array}{l} a_n = 5a_{n-1} - 6a_{n-2} + 3^n \\ a_n - 5a_{n-1} + 6a_{n-2} = 3^n \end{array}\right]$ $a_1 = 5$ $a_2 = 3$

$a_n^h = c_1 2^n + c_2 3^n$

$a_n^p = d \cdot 3^n$ [but merging/collision may happen, then resolve it]

$= dn \cdot 3^n$

$dn \cdot 3^n - 5d(n-1) \cdot 3^{n-1} + 6 \cdot d(n-2) \cdot 3^{n-2} = 3^n$

$9\cancel{n}d - 15\cancel{d}n + 15d + 6\cancel{d}n - 12d = 9$

$\cancel{-27d = 9}$ $3d = 9$

$\boxed{\cancel{d = -1/3}}$ $\boxed{d = 3}$

$a_n = c_1 2^n + c_2 3^n + 3n3^n$

$= c_1 2^n + c_2 3^n + n \cdot 3^{n+1}$

Ques $a_n = 5a_{n-1} - 6a_{n-2} + n \cdot 5^n$

$a_n^h = c_1 2^n + c_2 3^n$

$a_n^p = (d_0 + d_1 n) 5^n$

$\therefore (d_0+d_1 n)5^n - 5(d_0+d_1(n-1))5^{n} + 6(d_0+d_1(n-2))5^n = 5^n$

$25d_0 + 25d_1 n - 25d_0 - 25d_1 n + 25d_1 + 6d_0 + 6d_1 n - 12d_1 = 25$

<u>Ques</u> <u>LRVC to LRCC</u> -

$n a_n = 5n a_{n-1} - 5a_{n-1} + 2$, with $a_1 = 5$

↳ LRVC (variable coefficient)

$n a_n = 5(n-1)a_{n-1} + 2$

- Every LRVC, Indeterminate is not possible to solve only some of them can be solved.

$n a_n - 5(n-1)a_{n-1} = 2$ Let $\boxed{b_n = n a_n}$

$b_n - 5b_{n-1} = 2$

$a_n = a_n^h + a_n^p$

$b_n^h = t - 5 = 0 \quad t = 5$

$b_n^h = c_1 \cdot 5^n$

$b_n^p = d$

$d - 5d = 2$

$-4d = 2 \Rightarrow \boxed{d = -1/2}$

$\therefore b_n = c_1 \cdot 5^n + (-1/2)$

$b_n = c_1 5^n - 1/2$

$b_1 = 1 \cdot a_1$

$\boxed{b_1 = 5}$

$5 = c_1 5 - 1/2 \quad c_1 = 11/5$

$b_n = \frac{11}{5} 5^n - \frac{1}{2}$

$\therefore a_n \cdot n = \frac{11}{5} 5^n - \frac{1}{2}$

$$\left\{ a_n = \frac{\frac{11}{5} \cdot 5^n - \frac{1}{2}}{n} = \frac{1}{n}\left[\frac{11}{5} \cdot 5^n - \frac{1}{2}\right] \right\}$$

Ques $a_n^2 = a_{n-1}^2 + 2 \quad a_1 = 5$

$a_n^2 - a_{n-1}^2 = 2$

Let $b_n = a_n^2$

$b_n = b_{n-1} + 2$

$b_n - b_{n-1} = 2$

$b_n = b_n^h + b_n^p$

$b_n^h = t - 1 = 0$

$t = 1$

$b_n^h = c_1$

$b_n^p = d \cdot n$

$dn - d(n-1) = 2$

$\boxed{d = 2}$

$\Rightarrow b_n = c_1 + 2n$

$\therefore a_n = \sqrt{b_n}$

$a_n = \pm\sqrt{c_1 + 2n}$

- it will be positive as a_1 is positive

$a_n = \sqrt{c + 2n}$

$5 = \sqrt{c + 2 \cdot 1}$

$c = 23$

$\boxed{a_n = \sqrt{23 + 2n}}$

Ques- $\cancel{a_n^2 =}$ $a_n^2 = 8a_{n-1} \quad a_1 = 5$

take log both sides,

$2\log a_n = \log 8 + \log a_{n-1}$

$2\log a_n = 3 + \log a_{n-1}$

Let $b_n = \log a_n$

$2b_n - b_{n-1} = 3$

$b_n = b_n^h + b_n^p$

1) $b_n^h \Rightarrow 2t - 1 = 0$

$t = 1/2$

2) $b_n^h = c(1/2)^n$

$b_n^p = d$

3) $2d - d = 3$

$\boxed{d = 3}$

$\boxed{b_n = c(1/2)^n + 3}$

$b_n = \log a_n \Rightarrow a_n = 2^{b_n}$

$a_n = 2^{[c(1/2)^n + 3]}$

put $n=1$, $5 = 2^{[c/2 + 3]}$

$\log 5 = c/2 + 3 \Rightarrow c = 2(\log 5 - 3)$

Indeterminate order to Determinate order.

Ques - $a_n = 5a_{n/3} + 6$ $[a_1 = 5]$ b.H should be polynomial.

Let $n = 3^k$

$\frac{n}{3} = 3^{k-1}$

$a_{3^k} = 5a_{3^{k-1}} + 6$

Let $b_k = a_{3^k}$

$b_k = 5b_{k-1} + 6$

$b_k = b_k^h + b_k^p \Rightarrow b_k^h \Rightarrow b_k - 5b_{k-1} = 0$

$\boxed{b_k^h = c_1 5^k}$

$b_k^p = d$

$\boxed{b_k^p = -3/2}$

$d = 5d + 6$

$-4d = 6$

$\boxed{d = -3/2}$

$\therefore b_k = c_1 \cdot 5^k - \frac{3}{2}$

$\therefore a_{3^k} = c_1 \cdot 5^k - \frac{3}{2}$ $k = \log_3 n$

$\boxed{a_n = c_1 \cdot 5^{\log_3 n} - \frac{3}{2}}$

put $a_1 = 5$ $5 = c_1 - 3/2$ $c_1 \Rightarrow 13/2$

$\boxed{a_n = \frac{13}{2} \times 5^{\log_3 n} - \frac{3}{2} = \frac{13}{2} \cdot n^{\log_3 5} - \frac{3}{2}}$

$a_n = 5a_{n/3} + 6 \cdot n^2$

$b_k = 5b_{k-1} + 6\ 3^{2k}$

$\boxed{= 5b_{k-1} + 6 \cdot 9^k}$

Set Theory

Set Theory

- Relations *** } Most Imp.
- Functions **
- Group Theory *
- ~~Graph Theory~~ POSET, LATTICES & Boolean Algebra

Relations:

1. Definition of Relation
2. R- Relative Set: (Domain & Range)
3. Representation of a Relation (8 ways)
4. operations on Relations
   [$R \cup S$, $R \cap S$, $R-S$, R^C, $R \oplus S$, $S-R$], R^{-1}, S^{-1} [$R \circ S$, $S \circ R$]. (7 operations) Inverse, Composition
5. Types of Relations **
   Reflexive [R, IR, S, AS, ATS, T] [6 Properties]
6. Counting of Relations & Closure of Relations IMP
7. Equivalence Relations & Partial Order Relation.
   [R, S, T] · used for grouping
   [R, ATS, T] · used for ordering
8. Properties of Equivalence Relations
   · Quotient Set - [A/R] (set, Relation)
9. Power of a relation (R^n)

Relation- $\boxed{\text{Relation from A to B} \subseteq A \times B}$

- Set of ordered pair

$A \times B = \{(x,y) \mid x \in A, y \in B\}$ Largest possible relation

$R = \phi$ (Smallest Relⁿ)

$E = \{e_1, e_2, e_3\}$ $C = \{C_1, C_2\}$

$E \times C = \{(e_1, c_1), (e_1, c_2) (e_2, c_1) (e_2, c_2), (e_3, c_1) (e_3, c_2)\}$

$R = \{(e_1, c_1) (e_1, c_2) (e_2, c_1) (e_3, c_1)\}$

Ex- $A = \{a, b, c\}$ $B = \{1, 2\}$

$A \times B = \{(a,1) (a,2) (b,1) (b,2) (c,1) (c,2)\}$

- $|A \times B| = |A| * |B|$

- Relation from A to B then A, B ⇒ parent set

- $\overline{R_1} = (A \times B - R_1)$

  $A \times B$ - Universal Relation from A to B

- Relation on A ⇒ on $A \times A$

$T = \{(x,y) \mid y = x^2\}$ by default it will be considered on $R \times R$, if not mentioned [Set of Real No]

i. $A \times B = B \times A$ False
ii. $|A \times B| = |B \times A|$ True
iii. $A \times A = A \times A$ True

$(A \times B) \times C = (A \times B \times C)$

- A - set of n element   B - set of m elements
  - $|A \times B| = mn$ element
  - No of relation from A to B $= 2^{mn}$ [Binary Relations]
  - Smallest possible Relation, $R_S = \phi \Rightarrow |R_S| = 0$
  - Largest possible Relation, $R_L = A \times B \Rightarrow |R_L| = mn$

- $(a,1) \in R \equiv aR1$
- $(x,y) \in R \equiv xRy$
- $(x,y) \notin R \equiv x \not{R} y$
- $(x,y) \notin R \equiv x \not{R} y$

Ternary Relation- $A = \{a,b\}$ $B = \{1\}$ $C = \{\alpha\}$

$A \times B \times C = \{(a,1,\alpha)\ (b,1,\alpha)\}$

R- Relative Set: Let $A = \{1,2,3\}$

R on A, $R = \{(1,1)(1,2)(2,1)(2,3)(3,1)\}$

- $R(1) = \{1,2\}$

$\forall x \in A$, R-Relative Set $R(x) = \{y \mid (x,y) \in R\}$

$R(2) = \{1,3\}$

$R(3) = \{1\}$

for $R: A \to B$, $\boxed{\text{R-Relative Set } R(x) \subseteq B}$

- R Relative Set can also be calculated for a set
- Domain for $R: A \to B$

$$\text{Domain}(R) = \{x \mid (x,y) \in R\} \subseteq A$$

Domain$(R) = \{1,2,3\}$

- Range for $R: A \to B$

$$\text{Range}(R) = \{y \mid (x,y) \in R\} \subseteq B$$

Range$(R) = \{1,2,3\}$

- Domain & Range are special type of Relative set.

Ques $R = \{(x,y) \mid y = x^2\}$ on Real No

What is domain & Range of R.

Domain (R) = {Set of all Real No} = R

Range (R) = {Set of non-negative Real No}

$= R^+ \cup \{0\}$

for Range $\Rightarrow x = \sqrt{y}$

$\boxed{\text{only ~~these~~ non-negative real no can be taken.}}$

Ques $T = \{(x,y) \mid y = 3x+1\}$ on $R \times R$

Domain(R) = Real No

Range(R) = $x = \dfrac{y-1}{3}$

∴ Set of Real No

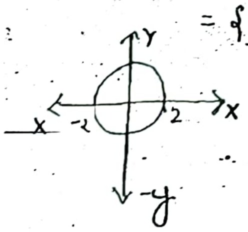
Ques $T = \{(x,y) \mid y = \dfrac{1}{x-2}\}$ on $R \times R$

Domain = {R} - {2}

Range = $x = \frac{1}{y} + 2$

= {R} - {0}

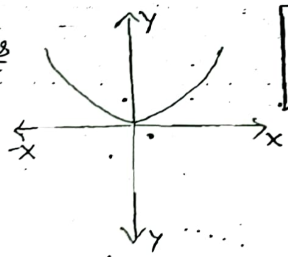
Ques



Domain(R) = $\{x \mid -2 \le x \le 2\}$

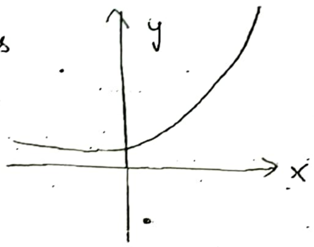
Range (R) = $\{y \mid -2 \le x \le 2\}$

Ques



$\boxed{\begin{array}{l}\text{Domain(R) = Real No} \\ \text{Range (R) = non negative Real No}\end{array}}$

Ques



$y = e^x$

Domain = R

Range = R^+

$x = \log y$

⇓

[(+)ve Real No]

$\boxed{\begin{array}{l} e^{-\infty} = 0 \\ \log 0 = -\infty \end{array}}$

Ques $A = \{1, 2, 3\}$ $R = \{\cancel{(a,1)}\ \cancel{(b,}\ (1,a)(2,b)(3,b)(3,a)\}$
$B = \{1, 2\}$

$R(B) = \{a, b\}$
↳ Relative set of B

- $\boxed{\text{Relative Set of } A,\ R(A) = \text{Range}(R)}$; $R(\text{Domain}(R)) = \text{Range}(R)$

these both of true bcoz for those element no element in range exist they will be ignored

- $R^{-1}(B) = \text{Domain}(R)$
  $R^{-1}(\text{Range}(R)) = \text{Domain}(R)$
  ↳ [Relative Set for the inverse relation]

- $\forall x\ R(R^{-1}(x)) = x$ [False] $R^{-1}(x) = y$ $R(y)$ need not be x
  It will be true in one to one Relation only
- $\forall x\ R^{-1}(R(x)) = x$ ~~[True]~~ [False]
  [Relative set]

  $R(x) = \{y, x\}$ $R = \{(x,x)(x,y)(y,z)(z,y)\}$
  $R^{-1}(R(x)) = \{x, y, z\}$

Representation of Relations :

1. LISTING Method
2. Set Builder
3. Statement
4. Matrix
5. Diagraph
6. Arrow diagram
7. Graph
8. Table

Imp

- if a relation is defined on single set than it should have square matrix

Let $R = \{(x,y) \mid x \le y\}$ on $A = \{1, 2, 3\}$

1) Listing - $R = \{(1,1)(2,2)(3,3)(1,2)(1,3)(2,3)\}$

2. Set Builder :

$R = \{(x,y) \mid x \le y\}$ on A

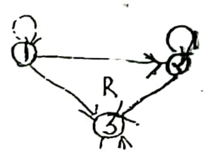
3) Statement :

xRy iff $x \le y$ on A

4) Matrix Rep: • for Relation on finite set

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

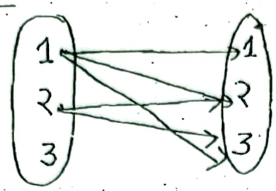
5. Diagraph - • can be drawn for rel^n on infinite set.
• must be designed for Relation on a single set.



◦ check for Reflexive : All self loop
Symmetric : two sided arrows

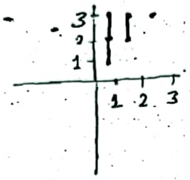
[All properties R, S, T, ATS, AS defined for Relⁿ on single set].

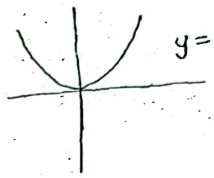
∴ Arrow Diagram:



- for finite set
- used in functions

∴ Graph ◦ mainly used for relation on uncountably infinite set





$y = x^2$

∴ Table · suitable for laboratory purpose

x	y
1	1
1	2
1	3
2	2
2	3
3	3

Operations on Relations:

Consider two Relations $R = \{(1,1)(1,2)(2,2)\}$ on set $A = \{1,2,3\}$
$S = \{(3,2)(2,2)\}$ on set $A = \{1,2,3\}$

1. $R \cup S = \{(x,y) \mid (x,y) \in R \text{ OR } (x,y) \in S\}$
$= \{(1,1)(1,2)(3,2)(2,2)\}$

2. $R \cap S = \{(x,y) \mid (x,y) \in R \text{ and } (x,y) \in S\}$
$= \{2,2\}$

3. $\bar{R} = \{(x,y) \mid \bar{R}\,(x,y) \notin R\} = A \times A - R$
$= \{(1,3)(2,1)(2,3)(3,1)(3,2)(3,3)\}$

On Relation R with n pairs on A with m element

$\boxed{|\bar{R}| = m^2 - n} \Rightarrow |\bar{R}| = |A|^2 - |R|$

4. $\bar{S} = \{(1,1)(1,2)(1,3)(2,1)(2,3)(3,1)(3,3)\}$

◦ $|\bar{R}| \neq |R|$

• $|R^{-1}| = |R|$ [TRUE]

Ques - Consider a relation as $|\bar{R}| = |R|$ on set A with 10 elmt. What is $|R|$?

$$|R| + |\bar{R}| = 10 * 10$$

$$\boxed{|R| = 100/2 = 50}$$

5> $R - S = \{(x,y) \mid (x,y) \in R \text{ and } (x,y) \notin S\}$

$$\left[\begin{aligned} R - S &= R \cap \bar{S} \\ &= R - (R \cap S) \end{aligned}\right]$$

$R - S = \{(1,1)(1,2)\}$

6> $S - R = \{(3,2)\}$

7. $R \oplus S = \{(x,y) \mid [(x,y) \in R \text{ and } (x,y) \notin S] \text{ or } ((x,y) \in S \text{ and } (x,y) \notin R)\}$

$= (R-S) \cup (S-R)$

$= (R \cup S) - (R \cap S)$

$= \{(1,1)(1,2)(3,2)\}$

$|R \oplus S| = |R| + |S| - 2|R \cap S|$

$\therefore |R \oplus S| = |R \cup S|$ iff R, S are disjoint i.e $|R \cap S| = 0$

• $R - S \neq S - R$

• $|R-S| \neq |S-R|$ they are equal if same size of S & R

8. $\underline{R^{-1}} = \{(x,y) \mid (y,x) \in R\}$

$= \{(1,1)(2,1)(2,2)\}$

• $R \neq R^{-1}$

• $|R| = |R^{-1}|$

• $R = R^{-1}$ iff R is symmetric.

Ques - $|R| = 10$ $|S| = 5$

what is $|R \times (S^{-1})|$?

Since $|S| = |S^{-1}|$

$\therefore |R \times S| = 10 * 5 = 50$

9. Composition - $R \circ S = \{(x,z) \mid (x,y) \in R \text{ and } (y,z) \in S\}$

$S \circ R = \{(x,z) \mid (x,y) \in S \text{ \& } (y,z) \in R\}$

$R \circ S = \{(2,2)(1,2)\}$

$S \circ R = \{(3,2)(2,2)\}$

• $R \circ S \neq S \circ R$ [not commutative]

• $R \circ (S \circ T) = (R \circ S) \circ T$ [Associative]

$\{(2,2)(1,2)\}$

For two non empty relation R & S, $R \circ S$ or $S \circ R$ may be empty
- if $R \circ S$ is empty, $S \circ R$ may or may not be empty & vice versa.

<u>Note:</u> 1. $|R \cup S| \leq |R| + |S|$
$|R - S| \leq |R|$
$|R \cap S| \leq |R|$
$|R \oplus S| \leq |R| + |S|$

If $|R| = 10$, $|S| = 15$
$|R \cup S|$ = atmost 25
$|R \cap S|$ = atmost 10
$|R - S|$ = atmost 10
$|R \oplus S|$ = atmost 25

If M_R be the matrix for relation R & M_S be the matrix for Relation S
than
- $M_{R \cup S} = M_R \vee M_S$
- $M_{R \cap S} = M_R \wedge M_S$
- $M_{\overline{R}} = \overline{M_R}$
- $M_{R^{-1}} = M_R^{t}$
- $M_{R-S} = M_R \wedge \overline{M_S}$
- $M_{S-R} = M_S \wedge \overline{M_R}$
- $M_{S \oplus R} = M_S \oplus M_R$

- $M_{R \circ S} = \underbrace{M_S \odot M_R}_{\text{boolean Mul.}}$
- $M_{S \circ R} = M_R \odot M_S$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

~~$M_R = M_R$~~
$M_R = M_R^{t}$ iff R is symmetric

Types of Relation:

1. **Reflexive**: for Relation R on A

$\forall x \in A,\ (x,x) \in R$ ie xRx

$A = \{1,2,3\}$ $R_1 = \{(1,1)(2,2)(3,3)(3,1)\}$ Reflexive

$R_2 = \{(1,1)(2,2)(2,3)(3,2)\}$ ~~IRRe~~ Not Reflexive

	R	IR	S	ATS	AS	T
$\{(1,1)(2,2)(3,3)(3,1)\}$	✓	X	X	✓	X	✓
$\{(1,1)(2,2)(2,3)(3,2)\}$	X	X	✓	X	X	X
$\{(x,y) \mid x = y^2\}$ on R	X	X	X	✓	X	X
$\{(x,y) \mid x+y=10\}$ on Z	X	X	✓	X	X	X
$\{(x,y) \mid x \leq y\}$ on Z	✓	X	X	✓	X	✓
$\{(x,y) \mid x$ divide $y\}$ on Z	X {0}	X	X	X {0}	X	✓
$\{(x,y) \mid x \parallel y\}$ all lines	✓	X	✓	X	X	✓
$\{(x,y) \mid x \perp y\}$	X	✓	✓	X	X	X
$\{(x,y) \mid x$ is brother of $y\}$	X	✓	X	X	X	X $xRy \wedge yRx = xRx$ ↳ False
$\{(x,y) \mid x+y=9\}$ on Z	X	✓	✓	X	X	X

$y = mx$
$x = my$

2. **Irreflexive** $\forall x \in A \,.\, (x,x) \notin R$

- A reflexive relⁿ is always not a irreflexive relⁿ
- A IRReflexive relⁿ is not a reflexive

R ⇒ Not IR [True]

IR ⇒ Not R [True]

Not R ⇒ IR [False]

Not IR ⇒ R [False]

3. **Symmetric**: ~~$\forall x, y \in A$~~ if $(x,y) \in R$ than $(y,x) \in R$

$$\left[xRy \Leftrightarrow yRx \right]$$

4. **AntiSymmetric**: ~~$\forall x,y \in$ $xRy \not\Leftrightarrow yRx$~~ $xRy \Rightarrow y \not R x$ unless $x=y$

if xRy then $y \not R x$ unless $x = y$.

- if $(x,y) \in R$ than $(y,x) \notin R$ if $x \neq y$
  - Self loops are allowed.
- A Relation can be both symmetric as well as antisymmetric
  Ex- Self loops
- A relation Not Symmetric $\Leftrightarrow$ Antisymmetric [False]

S $\Rightarrow$ Not AS False
S $\Rightarrow$ AS False
AS $\Rightarrow$ S False
Not S $\Rightarrow$ AS False

- $\boxed{xRy \;\&\; yRx \Rightarrow x=y}$

Ex $= \{(x,y) \mid x<y\}$ → True Antisymmetric

Assymmetric- • A Relation is Assymmetric iff it is irreflexive & Antisymmetric

IR $\wedge$ ATS $\Leftrightarrow$ ATS

if $\boxed{xRy \Rightarrow y \not R x \wedge (x \neq y)}$

- Self loops are not allowed.

Ex - $\{(x,y) \mid x<y\}$
  ↳ Assymetric
  ↳ Antisymmetric
  ↳ Irreflexive

• $xRy \Rightarrow y \not R x$

- $\boxed{xRy \wedge yRx \Rightarrow \phi}$

Transitive- If $\boxed{xRy \wedge yRz \Rightarrow xRz}$

- No need to check self loop

Every Antisymmetric Relation is Assymetric False
Every ~~Assymetr~~ Asymmetric Relation is Antisymmetric True
∴ IR $\Rightarrow$ AS False
AS $\Rightarrow$ IR True

Note- 1. An Empty Relation R is always IR, S, ATS, AS, T.

Ques 1. $(x_1,y_1) R (x_2,y_2)$ iff $x_1 + y_1 = y_2 + x_2$
2. $(x_1,y_1) R (x_2,y_2)$ iff $x_1 \leq x_2$ & $y_1 \leq y_2$

Reflexive- 1. $(x_1,y_1) R (x_1,y_1)$
~~$x_1 + x_1 = y_1 + y_1$~~
∴ $x_1 + y_1 = x_1 + y_1$ (True)

2. $(x_1,y_1) R (x_1,y_1)$
$x_1 \leq x_1$
$y_1 \leq y_1$ True

Symmetric

$(x_1, y_1) R (x_2, y_2)$
$\Rightarrow x_1 + y_1 = x_2 + y_2$
$\therefore (x_2, y_2) R (x_1, y_1)$
$\Rightarrow x_2 + y_2 = x_1 + y_1$
True

$(x_1, y_1) R (x_2, y_2)$ $\quad (2, 3)\ (4, 5)$
$\Rightarrow x_1 \leq x_2$ & $y_1 \leq y_2$
then $(x_2, y_2) R (x_1, y_1)$
$x_2 \leq x_1$ & $y_2 \leq y_1$ False

Antisymmetric-

$(x_1, y_1) = (2, 3)$
$(x_2, y_2) = (4, 1)$
if $(x_1, y_1) R (x_2, y_2)$
$\Rightarrow x_1 + y_1 = x_2 + y_2$
$\Rightarrow x_2 + y_2 = x_1 + y_1$
$= (x_2, y_2) R (x_1, y_1)$
False

$\therefore$ Assymetric is also false

$x_1 \leq x_2$ & $y_1 \leq y_2$
then $x_2 \not\leq x_1$ $y_2 \not\leq y_1$
False
than $(x_2, y_2) R (x_1, y_1)$

- Assymetric is false because of being reflexive

Transitive

$(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3)$
$x_1 + y_1 = x_2 + y_2$
$x_2 + y_2 = x_3 + y_3$
$\Rightarrow x_1 + y_1 = x_3 + y_3$
$\Rightarrow (x_1, y_1) R (x_3, y_3)$
Hence it is transitive

$(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3)$
$(x_1 \leq x_2$ and $y_1 \leq y_2)$ & $(x_2 \leq x_3$ & $y_2 \leq y_3)$
$\Rightarrow x_1 \leq x_3$ &
$y_1 \leq y_3$
$\Rightarrow (x_1, y_1) R (x_3, y_3)$
Hence it is transitive

-Sept-2017

ounting of Relations:

No of Relations possible from a set $|A|=m$ to $|B|=n$ $\quad 2^{mn}$

Smallest Relation $=\phi$.

No of Relation ———————— $|A|=n$ $=2^{n^2}$

Largest Relation $= A\times A$
$= A\times B$

	No of relation	Smallest	Largest
Reflexive	2^{n^2-n}	ϕ n $I_A=[\text{Identity Rel}^n]=\Delta$	$\lvert A\times A\rvert=n^2$
Irreflexive	2^{n^2-n}	ϕ	$(A\times A-I_n)=n^2-n$
Symmetric	$2^{\frac{n^2+n}{2}}$	ϕ	$A\times A=n^2$
Antisymmetric	$2^n\cdot 3^{\frac{n^2-n}{2}}$	ϕ	$n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=\Sigma n$
Asymmetric	$3^{\frac{n^2-n}{2}}$	ϕ	$\frac{n^2-n}{2}=\frac{n(n-1)}{2}=\Sigma(n-1)$
Transitive	-	ϕ	$A\times A=n^2$
Equivalence	Procedure exist	$I_A=\Delta$	$A\times A=n^2$
POS	-	$I_A=\Delta$	$\frac{n(n+1)}{2}=\Sigma n$

Reflexive - Let $n=4$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & - & - & - \\ 2 & - & 1 & - & - \\ 3 & - & - & 1 & - \\ 4 & - & - & - & 1 \end{array} = 2^{n^2-n}$$

Diagonal element should be 1

IRReflexive

Diagonal's should be 0

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} - & 1 & - & - \\ 1 & - & - & - \\ 1 & 1 & - & - \\ 1 & 1 & 1 & - \end{bmatrix} = 2^{\frac{n^2-n}{2}}\cdot 2^n$$

2^n ↳ for diagonals

$2^{\frac{n^2-n}{2}}$ ↓ for symmetric pair ——— [either take both (pair) or don't take]

$$= 2^{\frac{n^2+n}{2}}$$

✓ 0 0
0 1
1 0
✓ 1 1

∴ No of boolean Matrix of size $n\times n$ which are symmetric $= 2^{\frac{n^2+n}{2}}$

4. Antisymmetric Diagonal element $= 2^n$

for Non Diagonal pair $= 3^{n^2-n/2}$. $[00, 01, 10]$

$\therefore$ $\boxed{\text{possible relation} = 2^n \cdot 3^{\frac{n^2-n}{2}}}$

Ratio of Symmetric Rel^n to Antisymmetric relation for n set

$n=3$

$$\frac{\cancel{2^n} \cdot 2^{\frac{n^2-n}{2}}}{\cancel{2^n} \cdot 3^{\frac{n^2-n}{2}}} = \left(\frac{2}{3}\right)^{\frac{n^2-n}{2}} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

5. Assymetric-

$$\begin{bmatrix} 0 & - & - & - \\ - & 0 & - & - \\ - & - & 0 & - \\ - & - & - & 0 \end{bmatrix}$$

$\Rightarrow$ Diagonal element $= 0$

$\therefore$ No of Diagonal $= 3^{\frac{n^2-n}{2}}$ $[00, 01, 10]$

$\therefore$ possible relations $= 3^{\frac{n^2-n}{2}}$

- Relation which are symmetric or Reflexive-

$n(R \cup S) = n(R) + n(S) - n(R \cap S)$

Identity $Rel^n = \{(x, y) \mid x = y\}$ on A

$= I_A$. • also represented by Δ

POR- [Reflexive, Antisymmetric, Transitive]

- $A \times A$ is guaranteed to be Equivalence relation
- Largest Antisymmetric rel^n is not unique. [either you can take upper triangle or the lower triangle]

Largest partial order relation

$$\begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{bmatrix}$$

$\therefore$ it is guaranteed to be transitive

if $(x, y) \in$ upper triangle
$(y, z) \in$ upper triangle
$(x, z) \in$ upper triangle

$\therefore$ No of ~~set~~ elements $= n + \frac{n^2-n}{2}$

$= \frac{n(n+1)}{2}$

flexive & Irreflexive (R∩IR) $\Rightarrow$ No of relation = 0 ,

lexive & Symmetric (R∩S)

$\Rightarrow$ No of relation = $2^{\frac{n^2-n}{2}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ \phi & 1 & 0 \\ \phi & 0 & 1 \end{bmatrix}$$

$2^n = 1$ $2^{\frac{n^2-n}{2}}$

flexive & Antisymmetric (R∩ATS)

$\Rightarrow$ No of Relation = $3^{\frac{n^2-n}{2}}$

lexive & Assymmetric (R∩AS)

$\Rightarrow$ No of Relation = 0 (bcoz Assymetric is always Irreflexive)

eflexive & Symmetric (IR∩S)

$\Rightarrow$ No of Relation = $2^{\frac{n^2-n}{2}}$

reflexive & Antisymmetric (IR∩ATS)

$\Rightarrow$ No of Relation = $3^{\frac{n^2-n}{2}}$

eflexive & Asymmetric (R∩AS)

$\Rightarrow$ No. of Relation = $3^{\frac{n^2-n}{2}}$

Diagonal element are 0

mmetric & Antisymmetric (S∩ATS)

$\Rightarrow$ No of Relation = 2^n [only self loops are allowed.

mmetric & Assymmetric (S∩AS)

$\Rightarrow$ No of Relation = 1

$R = \phi$

Since No self loop & no symmetric pair

$2^n = 1$ way $\quad 2^{\frac{n^2-n}{2}} - 1$ way

(no element taken) (no elmt taken)

$\therefore$ No of Relation = 1

ntisymmetric & Asymmetric (ATS∩AS)

$\Rightarrow$ No of relation = $3^{\frac{n^2-n}{2}}$

Note- By using inclusion-exclusion principle, all other ~~cases~~ cases (∪, ∩, -) can be done.

Closure of Relation:

1. Reflexive Closure.
2. Symmetric Closure.
3. Transitive Closure.

- All Closures are unique.

Reflexive Closure (RC): A relation T is RC for R iff

1) $R \subseteq T$
2) T should be reflexive
3) No subset of T should be reflexive.

i.e T must be the smallest relation containing R & reflexive.

Ex- $A = \{1, 2, 3\}$

$R = \{(1,1)(2,2)(3,2)(2,1)\}$

what is its reflexive closure?

$T_1 = \{(1,1)(2,2)(3,3)(3,2)(2,1)\}$ Reflexive closure

$T_2 = \{(1,1)(2,2)(3,3)(3,2)\}$ Not a Reflexive closure $[R \not\subseteq T]$

$T_3 = \{(1,1)(2,2)(3,3)(3,2)(2,1)(1,4)\}$ Not a Reflexive closure $[T_4 \subset T_3$ is RC$]$

Symmetric Closure (SC): A Relation T is SC for R iff

1) $R \subseteq T$
2) T should be symmetric
3) No subset of T should be symmetric

$T = \{(1,1)(2,2)(3,2)(2,3)(2,1)(1,2)\}$

$T_1 = \{(1,1)(2,2)\underline{(3,3)}(3,2)(2,3)(2,1)(1,2)\}$

→ Not SC

Transitive Closure-

1. T should be transitive
2. $R \subseteq T$
3. No subset should be transitive

$R = \{(1,1)(2,2)(3,2)(2,1)\}$

$T = \{(1,1)(2,2)(3,2)(2,1)(3,1)\}$ ← TC

Ques $A = \{1,2,3,4\}$

$R = \{(1,1)\,(3,3)\,(2,1)\,(1,2)\,(3,2)\,(1,4)\}$

What is its TC?

first copy each pair of R

$T = \{(1,1)\,(3,3)\,(2,1)\,(1,2)\,(3,2)\,(1,4)\,(2,2)\,(2,4)\,(3,4)\,(3,1)\}$

• check for newly added elmt with all existing elmt.

Ques- $R = \{(x,y) \mid x < y\}$

1) $RC = \{(x,y) \mid x \leq y\}$

2. $SC = \{(x,y) \mid x<y$ ~~and~~ OR $y<x\} = \{(x,y) \mid x \neq y\}$

3. $TC = \{(x,y) \mid x<y\}$ → already transitive

on A,

Note- 1) $RC(R) = R \cup \Delta$

2) $SC(R) = R \cup R^{-1}$

3) $TC(R) = R^1 \cup R^2 \cup R^3 \cdots\cdots = R^{\infty} = R^+$ [connectivity Relation]

4) $R^0 = \Delta = I$

5. $R^0 \cup R^1 \cup R^2 \cup R^3 \cdots\cdots$ → Reflexive Transitive closure (R^*) [Reachability Relation]

Ques- $R = \{(x,y) \mid y = x+1\}$

1) $RC = \{(x,y) \mid y = x+1$ or $y = x\}$

2) $SC = \{(x,y) \mid y = x+1$ or $x = y+1\} \equiv \{(x,y) \mid y \neq x+1$ or $x = y+1\}$ — $y = (y-1)$ [$x = y \pm 1$]

3) $TC = \{(x,y) \mid y = x+z,\ z \in N\}$ or $y \geq x+1$

$TC = \{(x,y) \mid y = x+1$ or $y = x+2$ or $x+3 \cdots\cdots]$

R $\quad R^2 \quad R^3$

Ques- What is the transitive closure of R on set A, $|A| = 3$

a) $R^0 \cup R^1 \cup R^2 \cup R^3$

b) $R^1 \cup R^2 \cup R^3 \cup R^4 \cup R^5 \ldots\ldots$ [No new element will be added after R^3]

c) $R^1 \cup R^2 \cup R^3$

Note: $\boxed{\text{on } A,\ |A| = n \quad R^{n+1} \subseteq R^0 \cup R^1 \cdots \cup R^n}$

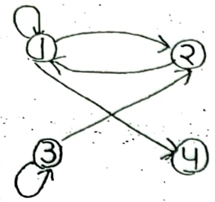
$R \circ S = \{(x,z) \mid (x,y) \in R\ (y,z) \in S\}$

$R \circ R = \{(x,z) \mid (x,y) \in R\ (y,z) \in R\}$

$\therefore (x,z)$ will be looked into R^2

2. $(x, y) \in R^{\infty}$ if and only if x & y are connected in diagraph of R.
[via directed path]

Ques- $R = \{(1,1)\ (3,3)\ (2,1)(1,2)\ (3,2)(1,4)\}$



1 length path — 2 length path — 3 length path

$TC(R) = \{(1,1)(1,2)\ (1,4),(2,1)(2,2)(2,4)$
$(3,1)\ (3,2)\ (3,3)\ (3,4)\}$

[take all the pairs which are reachable from a vertex.]

- it is known as connectivity Relation as it shows all the paths present in diagraph.
- it covers all the paths of length 1 or more.

~~Ques- R = f~~

Note- 1. self loops belongs to R^+.
2. there may be case all self loops belongs to R^+.
3. It may or may not contain self loops.
4. R^* is reachability relation as it already includes all the self loops, since a node is always reachable from itself but may not be connected.

Ques-. $R = \{(x,y) \mid y = x+1\}$

$R^* = \{R^0\} \cup \{TC\}$
$= \{(x,y) \mid y = x\} \cup \{(x,y) \mid y \geq x+1\}$
$= \{(x,y) \mid y \geq x\}$

EQUIVALENCE RELATION :

- Reflexive, Symmetric, Transitive
- grouping is done.
  Ex $= \{(x,y) \mid x \parallel y\}$
- equivalence classes are created & related elements are in same classes.
- Application: DBMS Query

- To put things in order - partial order relation
- for equivalence (same, parallel, equal)

$\leq$ $\{(x,y) \mid x = y\}$

- $\{(x,y) \mid x$ & y on same letter starting$\}$
- $\{(x,y) \mid |x| = |y|\}$
- $\{(x,y) \mid x \equiv y \bmod m\}$
- $(x_1, y_1) R (x_2, y_2) \quad x_1 + y_1 = x_2 + y_2$
- $(x_1, y_1) R (x_2, y_2) \quad x_1 = x_2$ and $y_1 = y_2$

$\leq$ $\boxed{(x_1, y_1) R (x_2, y_2) \quad x_1 = x_2 \text{ [OR] } y_1 = y_2}$

Reflexive- $(x_1, y_1) R (x_1, y_1) \quad x_1 = x_1$ OR $y_1 = y_1$ [True]

Symmetric : $(x_1, y_1) R (x_2, y_2) \quad x_1 = x_2$ OR $y_1 = y_2$

$\Downarrow$

$x_2 = x_1$ OR $y_2 = y_1$ [True]

Transitive $(x_1, y_1) R (x_2, y_2) \quad x_1 = x_2$ OR $y_1 = y_2$

$\begin{matrix} x_1 & y_1 \\ (1, & 2) \end{matrix} \quad \begin{matrix} x_2 & y_2 \\ (1, & 4) \end{matrix}$

$y_2 = y_3$ OR $x_2 = x_3$

$\begin{matrix} x_3 & y_3 \\ (4, & 4) \end{matrix}$

$\therefore$ False

$x_1 \neq x_3$ [OR] $y_1 \neq y_3$

Antisymmetric- False

$\boxed{\text{Hence it is not equivalence relation \& not a partial order relation}}$

Quotient Set: A Relation R on Set A & R is an equivalence Rel^n

$\boxed{\text{Quotient set } A/R = \text{partition set of } R}$

Theorem- If R is an equivalence relation than Quotient of R [A/R] is the partition of A, which is unique for R.

- Every equ. Relation corresponds to a unique partition. [one to one correspondence]

Partition Set for a set A of n element

Partition Π is given as

$\Pi = \{A_1, A_2 \dots A_n\}$ $A_1, A_2 \dots A_n \subseteq A$ [blocks]

$\Rightarrow$ Π is partition of A if

(i) $|A_i| \geq 1$

(ii) $A_i \cap A_j = \phi$ $i \neq j$ $\forall A_i, A_j \in \Pi$

(iii) $\cup A_j = A$

Ex 1) $\{\{\}, \{1,2,3,4\}\}$ → Not a partition

2) $\{\{1,2\} \{2,3\} \{4\}\}$ – Not a partition

3) $\{\{1,2\}, \{3,4\}\}$ – partition

4) $\{\{1,2\}, \{3\}\}$ – Not a partition.

5) $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ – partition

6) $\{\{1,2,3,4\}\}$ – partition

What is the valid partition of Z

(a) $\{\{x \in Z \mid x \geq 1\} \cup \{x \in Z \mid x \leq 1\}\}$

(b) $\{\{x \in Z \mid x \geq 1\} \cup \{x \in Z \mid x < 1\}\}$ ✓

(c) $\{\{x \in Z \mid x > 1\} \cup \{x \in Z \mid x < 1\}\}$

(d) $\{\{x \in Z \mid x \geq 1\} \cup \{x \in Z \mid x \ngeq 2\}\}$

Quotient Set- • Set of equivalence classes.

$R = \{(1,1) (1,2) (2,1) (2,2) (3,3)\}$ on $A = \{1,2,3\}$.

$\boxed{[A/R] = \text{Set of equivalence class of every element of } A}$

$\{1, 2\}$
$\{(1,1)(1,2)$
$(2,3)$

$[1]$ = Relative set of 1 $= \{1, 2\}$
$[2]$ = Relative set of 2 $= \{1, 2\}$
$[3]$ = Relative set of 3 $= \{3\}$

- $[A/R] = \{[1], [2], [3]\}$
  ↳ Set of distinct equivalence classes.
  $= \{\{1,2\}\{3\}\}$

Note- • for a relation (equ^n) unique partition exist
• for a partition, a unique relation exist.
• No of equivalence possible of a set of size n = No of partition possible for a set of size n.
• Quotient set A/R is the partition corresponding to the relation R.

Ques- A partition Set $= \{\{1, 2\}, \{3\}\}$
What is R?

Ans- Let $\pi = \{A_1, A_2, \ldots A_n\}$
$R = \{A_1 \times A_1 \cup A_2 \times A_2 \cup A_3 \times A_3 \cup \ldots A_n \times A_n$
$= \{(1,1)(1,2)(2,1)(2,2)(3,3)\}$

obtain Relation R from π

Note- Partition set $\pi = \{A_1, A_2 \ldots A_n\}$
for A of size n $\quad |A_1| = 1 \; |A_2| = 1 \ldots |A_n| = 1$
then $\boxed{R = I_A}$

Ques- $W = \{$cat, catch, call, ball, bat$\}$
$R_2 = \{(w_1, w_2) \mid w_1, w_2$ Starts with same letter$\}$
What is W/R_2?

Ans- $W/R_2 = \{\{$cat, catch, call$\}\{$ball, bat$\}\}$

$R_1 = \{(w_1, w_2) \mid |w_1| = |w_2|\}$
$W/R_1 = \{\{$cat, bat$\}\{$call, ball$\}\{$catch$\}\}$

Index of Relation - • No of distinct Equivalence Classes

$R_1 = 3$

$R_2 = 2$

Ques- W = Oxford dictionary

1. $R = \{(w_1, w_2) \mid w_1, w_2$ start with same letter$\}$

Index of Relation = 26

2. $R_2 = \{(x,y) \mid x \equiv y \mod m\}$ - Congruence Modulo M

$\boxed{\text{Index of Relation} = M}$ {0 residue class, 1 residue class ..... m-1 residue class}

Note-1. If Index of Relation R on set A with m elements is m than $\boxed{R = I_A}$

$1 \leq \text{Index} \leq |A|$

Properties of Equivalence Classes-

for $E_1, E_2 \ldots E_n$ on A

1. $E_1 \cup E_2 \cup E_3 \ldots E_n = \cup E_i = A$ [Transitive]
2. $|E_i| \geq 1$ bcoz every element belongs to its own equivalence class $a \in [a]$ • [Reflexive]
3. $E_i \cap E_j = \phi$ if $i \neq j$ (Symmetry)
4. $\cap E_i = \phi$

Theorems 1. R is equivalence relation $\Leftrightarrow$ R^{-1} is equivalence relation

$\boxed{\text{Dual \& inverse are related}}$

- Self loops will remain same Reflexive
- $(x,y) \rightarrow (y,x)$, $(y,x) \rightarrow (x,y)$ } Symmetry
- $(x,y) \rightarrow (y,x)$, $(y,z) \rightarrow (z,y)$ $\Rightarrow (z,x)$ } Transitive

- Reflexive, Symmetry, transitive is closured ~~under~~ inverse.
- Partial order Relation is also closure under inverse.
- Antisymmetry, Asymmetry is also closed under inverse.

ote- 1) Dual of $A = A^*$ AND ⟺ OR
TRUE → FALSE

2. ~~complement~~ $\boxed{A^*(\bar{p}, \bar{q}) = \neg(A(p,q))}$

3. $(A^*)^* = A$

orem - If R & S be two equivalence relation on A
than (may or may not be)
1) RUS is ~~an~~ surely an equivalence relation. False
2) $R \cap S$ is surely an equivalence relation. True
[one way theorem]

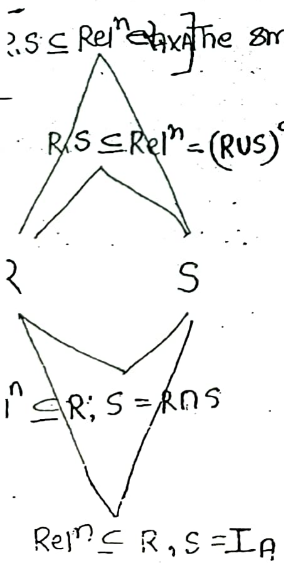
RUS - 1) Reflexive : True
2) Symmetric : True
3) Transitive : False
$R = \{(1,2)(2,3)(1,3)\}$
$S = \{(2,4)(4,5)(2,5)\}$
$R \cup S = \{(1,2)(2,3)(1,3)(2,4)(4,5)(2,5)\}$
Not a transitive Rel^n as (1,4) has to be added.

orem- for two Equivalence relation R & S.
1. the Largest Equivalence Relation which is inside R & S
$Eq \subseteq R, S$.

$$\boxed{\text{Largest Equivalence Relation} \subseteq R, S = R \cap S}$$

R,S ⊆ Rel^n ~~AxA~~ The smallest Equivalence relation contains R & S

$$\boxed{R, S \subseteq \text{Smallest Equivalence relation} = (R \cup S)^\infty}$$
→ to satisfy transitive closure



$$\boxed{\text{Smallest equivalence relation} \subseteq R, S = I}$$

It can't be RUS bcoz RUS may or may not satisfy transitive property

$$\boxed{\text{Largest Equivalence rel}^n \text{ which contains } R, S = A \times A}$$

Powers of a Relation (R^n):

$R^0, R^1, R^2, \ldots$

$R^{-1}, R^{-2}, \ldots$

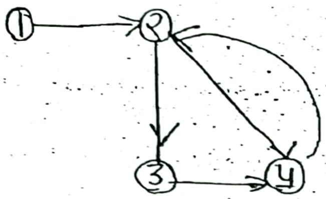
for Relation R on A

- $R^0 = I_A$
- $R^1 = R$
- $R^2 = R \circ R$ [all paths of length 2]
- $R^3 = R \circ R^2$ [all path of length 3]
- $R^{-1} = R^{-1}$
- $R^{-2} = R^{-1} \circ R^{-1}$ [$(x,y) \in R^{-2}$ if path of length -2 exist from x to y]

OR

- $R^{-2} = (R^2)^{-1}$ [$(x,y) \in R^{-2}$ if path of length 2 exist from y to x]

- $\boxed{R^m \circ R^n = R^n \circ R^m}$
- $(R^m)^n = R^{mn}$



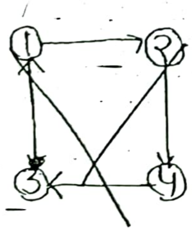
$R^{-2} = \{ (3,1) (4,1) (4,2) (2,2) (2,4) (3,4) (2,3)$ ~~(3,4)~~ $(4,4) \}$

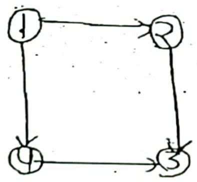
Ques- $M_{R^3} = \begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} & \\ — & 5 \\ & \end{bmatrix} \end{array}$ ⇒ [there exist exactly 5 path of length 3 b/w i & j]

- Matrix will tell about exactly No of path along with length of path.

- if instead of 5, 1 is given then we are not sure whether mat. multiplication or boolean Multiplication than we choose atleast 1.

Ques - for the given diagraph for M_{R^2} what is $M_{R^2}(1,3)$





$M_{R^2}(1,3) = 2$

[exactly 2 path exist]

-sept-2017

unctions :

- Definition, Domain & Range
- Types of functions
  - (i) Mapping
  - (ii) partial
  - (iii) one to one
  - (iv) Many to one
  - (v) Into
  - (vi) Onto
  - (vii) Bijection
  - (viii) PerMutation f^n
- Counting of functions
- Inverse of function (f^{-1})
- composition fog

efinition - A Relation in which every Input has a Unique Output.

on Set $A = \{1, 2, 3\}$ $B = \{a, b, c\}$

$S : A \to B$ $\quad S = \{(1,a)(2,b)(3,b)(2,c)\}$

— Not a function as 2 outputs b as well as c.

$S' = \{(1,a)(2,b)(3,b)\}$ → Function

In Listing Method, a set is a function if first element never repeats in set.

Q. Which of the following is not a f^n? on $R \times R$

a) $\{(x,y) \mid y = x^2\}$

b) $\{(x,y) \mid y = \sqrt{x}\}$ $\quad$ if $x = 4$ $\{(4,2)(4,-2)\}$ [it can be made f^n by selecting either (+)ve or (-)ve value]

(c) $\{(x,y) \mid y = \sin x\}$

(d) $\{(x,y) \mid y = e^x\}$

e) $\{(x,y) \mid y = \sin^{-1} x\}$ $\quad x = 0 \quad y = \pm n\pi$ [by choose the range as $-\pi/2$ to $\pi/2$ [principal value]]

(f) $\{(x,y) \mid y = |x|\}$

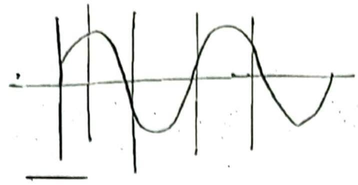
Graph Method -
- Draw the graph
- Draw vertical lines on graph
- if vertical line cut graph only once then f^n otherwise it is not.

$y=x^2$

f^n

$y=\sqrt{x}$

Not a f^n



$y=\sin x$

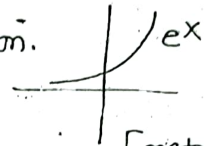
[Function]

$y=\sin^{-1}x$

[not a f^n]

to get inverse of a f^n, flip its graph by $90°$

- If horizontal line also cut it once, then one-to one function.



[one to one f^n]

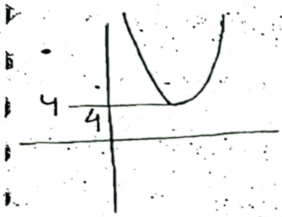
$\{(x,y) \mid y=\sqrt{x}\}$

- Domain $= \{R^+\} + \{0\}$
- Range $= R$

$\{(x,y) \mid x^2+y^2=4\}$

Domain $= x \in R \ -2 \leq x \leq 2$

Range $= R$



Domain $= R$

Range = all Real No greater than 4

Types of function:

1. Mapping / Entire f^n / total f^n : A function f is a mapping $f: A \to B$

iff Domain$(f) = A$

- It is denoted by $f_{A\to B}$ $\Rightarrow$ [Domain $=A$]

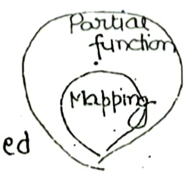
Consider the f^n $f: R \to R$

1) $f(x) = x^2$ Domain $= R$
2) $f(x) = \log x$ Domain $= R^+$ Not a Mapping [but a partial function]
3) $f(x) = e^x$ Domain $= R$
4) $f(x) = \sin x$ Domain $= R$

2. Partial function - Every f^n is a partial f^n.

Domain $\subseteq A$

i.e Some element of A is not related to any element of B.



Every mapping is a partial function
onsider the arbitrary given function as mapping

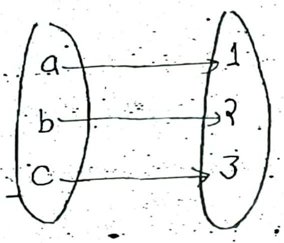
e to one function ;

Domain → Codomain

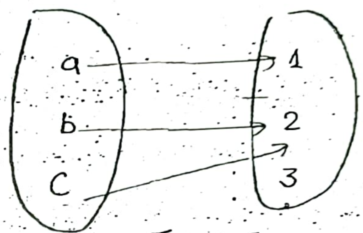
nto function- A function $f : A \to B$ is onto
[Surjection] iff $\boxed{\text{Range}(f) = \text{codomain}(f) = B}$

i.e every element of B is related to some element of A.

Into function- A f^n $f : A \to B$ is into
iff $\boxed{\text{Range}(f) \subset \text{codomain}(f) \subset B}$



[Onto]



[Into]

A f^n $f : A \to B$ is onto
$\forall x \in B \; \exists y \in A$ such that $f(y) = x$

A f^n $f : A \to B$ is into.
$\exists y \in B \; \forall x \in A$ such that $f(x) \neq y$

les- $f(x) = \begin{cases} 0 & x \text{ is odd} \\ 1 & x \text{ is even} \end{cases}$ on $Z \times Z$

- many to one f^n.
- into f^n $Z \to \{0, 1\}$
- if $f(x)$ on $Z \times \{0,1\}$ ....
- onto f^n

es $f(x) = 3x+1$ on $Z \times Z$
→ into it never generate some of value {0 etc}

- one to one
$3x_1 + \cancel{1} = 3x_2 + \cancel{1}$
$\boxed{x_1 = x_2}$

Ques- $f(x) = \log x$ on $R^+ \to R$

$y = \log x$

$x = e^y$

Here Range $= R$ Hence onto

Let $f(x_1) = f(x_2)$

$\log x_1 = \log x_2$

$x_1 = x_2$

Hence one to one

Ques $f(x) = e^x$ on $R \times R$

$y = e^x$

$x = \log y$ Range $= R^+$ Hence into

Let $f(x_1) = f(x_2)$

$e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$

~~one~~ one to one

Ques $f(x) = x^2$ on $R \to R$

Range $= \{R^+\} \cup \{0\}$

Hence into

Let $f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$(x_1 + x_2)(x_1 - x_2) = 0$

$\boxed{x_1 = \pm x_2}$. not one to one

~~Onto~~ One to One: on f^n $A \to B$

$f: A \to B$

$$\left\{ \text{If } f(x_1) = f(x_2) \Rightarrow \boxed{x_1 = x_2} \right\}$$

Ques $f(x) = x^2$ on $R^+ \to R$

Let $f(x_1) = f(x_2)$

$x_1^2 - x_2^2 = 0$

$(x_1 - x_2)(x_1 + x_2) = 0$

$x_1 = \pm x_2$

Not possible

$\to$ Hence one to one & onto

Ques $f(x) = \sin x$ on R to R

$\sin 0 = 0$

$\sin \pi = 0$

$\Rightarrow$ Hence it is not one to one

$\sin x \in [-1, 1] \subset R$

Hence it is not Onto

Neither bijection nor surjection.

Ques $f(x) = 3x + 1$ on $R \to R$

$f(x_1) = f(x_2)$

$3x_1 + \cancel{1} = 3x_2 + \cancel{1}$

$\boxed{x_1 = x_2}$

Hence one to one

$3x + 1 \in R$ $x = \frac{y-1}{3}$

for $\forall x \in R$

Hence onto

$\boxed{\text{Bijection as well as surjection}}$

Ques $f(x) = x^3$ on R to R

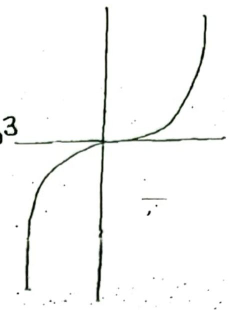
Let $f(x_1) = f(x_2)$

$x_1^3 = x_2^3$

three soln

$x_1 = x_2$, $x_1 = x_2 \omega$, $x_2 = \omega^3$

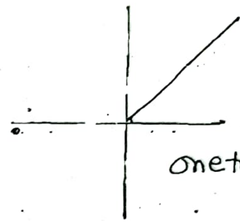
Hence one to one on R → R



Ques- $f(x) = |x|$ on R to R

not a onetoone fn

Ques $f(x) = |x|$ on $R^+ \to R$



onetoone.

Bijection- • A function f is bijection iff it is one to one and onto.

• one to one corresponcence function. • it should be a mapping

Theorem- If A function is bijection then cardinality of A is same as of B

$f : A \to B$ is bijection $\Rightarrow |A| = |B|$

$\boxed{\text{• There exist a bijection } f : A \to B \text{ if and only if } |A| = |B|}$

• There exist a bijection $f : A \to B$ ~~if & only if~~

• if A & B are infinite ⟺ if cardinality of both A & B is CI or UCI.

↓

[alpha ∅]

{1, 2, 3, 4, 5} {a, b, c} ~~f~~{(1, a) (2, b) (3, b)} - partial fn

Ques is there a bijection b/w set of RE language & Σ^*

↓ CI ↓ UCI → Not possible.

CFL language & Regular language

⇓ CI ⇓ CI - possible

is there a bijection possible for any two languages? → False

Both will be CI but one may be finite or infinite

- bijection b/w two finite languages - True
- bijection b/w two infinite language - False

Permutation function - A $f: A \to A$ is bijection is a permutation.
A bijection onto itself.

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

- there always exist a bijection in permutation function.

Counting of Functions: $f: A \to B$

• No of function possible $\lvert A\rvert = m \to \lvert B\rvert = n$ (Total)	n^m 1 2 3 .... m n ways
• partial function possible	$(n+1)^m$ $(n+1)$ ways ⇒ assign or not assign
• partial f^n but not total f^n	$(n+1)^m - n^m$
• one to one f^n	^nP_m
• Many to one f^n	$n^m - {}^nP_m$
• bijection f^n	$n!$
• permutation f^n	$n!$
• onto f^n	
• into f^n	

Partial f^n

1 2 3 ... m

$\Rightarrow (n+1)^m$ ways

$(n+1)$ ways - either assign (n ways)
[OR] ⇒ $n+1$ ways
- dont assign (1 way)

one to one f^n:

1 2 3 4 . . m

- we basically consider them as mapping

n way, $(n-1)$, $(n-2)$ — $(n-(m-1))$

$$f^n = n*(n-1)*(n-2)(n-3)\dots(n-(m-1)) = \frac{n!}{(n-m)!}$$

Bijection- $|A| = n \quad |B| = n$ [Because bijection is possible when elmt are equal]

$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & n \\ n & n-1 & n-2 & & n-(n-1) \end{array}$$

1 way

$\Rightarrow$ Total f^n possible
$= n * (n-1) * (n-2) \dots 1$
$= n!$

Permutation-

$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & n \\ n & n-1 & n-2 & & 1 \end{array}$$

$\Rightarrow \boxed{\text{Total } f^n \text{ possible} = n!}$

~~Into f~~

~~Into function~~-

Ques- Let S be set of all f^n

$S = \{f_1 \dots f_n\}$

on $\{0,1\}^3 \rightarrow \{0,1\}$

If $f: S \rightarrow \{0,1\}$

What is the No of possible f^n?

$|S| = 2^8$

$|f| = 2^{|S|} = 2^{2^8} = \underline{\underline{\cancel{256}\ 2^{256}}}$

Ques- Let No of function possible from $|X| \rightarrow |Y|$ is 91

What is the cardinality for $|X|, |Y|$

$91 = |Y|^{|X|}$

$\Rightarrow |X| = 1 \quad |Y| = 91$

$\cancel{(2^a)^b}$

$\cancel{2^{a^b}}$

Into function-

- $|A| = m \quad |B| = n$ element
- atleast one of into left ~~an~~ unassigned.

$n(1 \cup 2 \cup 3 \cup \dots n) \quad = n(1) + n(2) + \dots n(n) - n(1 \cap 2) \dots$

1: one left unassigned
2: two left unassigned
n: n left unassigned

$= {}^nC_1 (n-1)^m - {}^nC_2 (n-2)^m + {}^nC_3 (n-3)^m + \dots + {}^nC_n (n-n)^m$

one term is left unassigned; two term are left unassigned; 3 left unassigned.

$= {}^nC_1 (n-1)^m - {}^nC_2 (n-2)^m + {}^nC_3 (n-3)^m \dots {}^nC_{n-1} 1^m$

$$= \sum_{r=1}^{n-1} (-1)^{r-1}\, {}^nC_r (n-r)^m$$

from $|A| = 4$ to $|B| = 3$ How many Into f^n are possible?

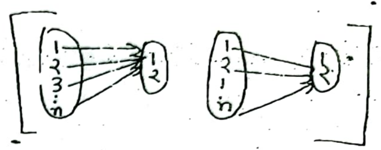
Sol- No of into f^n $= {}^3C_1 (2)^4 - {}^3C_2 (1)^4$

$= 3 * 16 - 3 = \underline{45}$

Ques No of into f^n possible from $|A| = n$ to $|B| = 2$

No of into $f^n = {}^2C_1 (2-1)^n - {}^2C_2 (2-2)^n$

$= {}^2C_1 = 2$



No of onto f^n = total f^n − into f^n

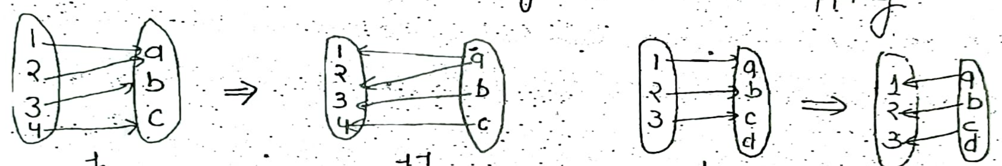
$= 2^n - 2$

Inverse (f^{-1}) : A function f is invertible iff f is bijection.

bcoz

if not one to one ⇒ f^{-1} will not be a function

if ~~on~~ not ~~onto~~ ⇒ f^{-1} will not be a mapping



f [f not one to one]

f^{-1} Not a function

f [not an onto f^n]

f^{-1} [not a mapping]

Theorem: f is bijection iff f^{-1} is a bijection.

Ques- $f(x) = 3x+1$ on $R \times R$ — Let $f(x) = y$ ⇒ $\cancel{f^{-1}(f(x))}$ $x = f^{-1}(y)$

$f^{-1}(x) = ?$

$y = 3x+1$

$x = \frac{y-1}{3}$ ∴ $\boxed{f^{-1}(x) = \frac{x-1}{3}}$

Ques $f(x) = e^x$ on $R \times R$

$f^{-1}(x) = ?$ $y = f(x)$

$y = e^x$

$\boxed{f^{-1}(x) = \log_e x}$ $x = \log_e y$

Ques- $f(x) = 3x^3 - 4x^2 + 5x + 1$

What is $f^{-1}(x)$?

It is quite difficult to get the result by conversion.

So use brute force method

put $x=1$ $f(x)=2$ then put 2 in option & if get 1 (option will be answer)

Ques $f(x,y) = (x+y, x-y)$

What is $f^{-1}(x,y)$

$f(x,y) = (x+y, x-y) = (x_1, y_1)$

$x_1 = x+y$

$y_1 = x-y$

$x = \frac{x_1+y_1}{2}$ $\quad y = \frac{x_1-y_1}{2}$

$$\boxed{f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)}$$

Rigourous method

$f(1,1) = (2,0)$

$f(2,3) = 5, -1$

(a) $(x-y, x+y)$ $\quad f(2,0) \Rightarrow (2,3)$

(b) $(x-2y, x+2y)$ $\quad f(2,0) \Rightarrow (2,2)$

(c) $\left(\frac{x-y}{2}, \frac{x+y}{2}\right)$ $\quad f(2,0) \Rightarrow (1,1)$ $\quad \cancel{f(2,3) = \left(\frac{1}{2}, \frac{5}{2}\right)}$ $f(5,-1)$ $(3,2)$

✓(d) $\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ $\quad f(2,0) \Rightarrow (1,1)$ $\quad f(5,-1) = (2,3)$

Properties-

1. $f = (f^{-1})^{-1}$
2. $f \circ f^{-1} = f^{-1} \circ f = I_A$

Composition of two function

- $fog(x) = f(g(x))$

<u>Ques</u> $f(x) = 3x+1$

$g(x) = \sin^2 x$

$\Rightarrow fog(x) = f(g(x))$
$= f(\sin^2 x)$
$= 3\sin^2 x + 1$

$gof(x) = g(f(x))$
$= g(3x+1)$
$= \sin^2(3x+1)$

- $gof(x) = g(f(x))$
- $gof(x) \neq fog(x)$ [Not commutative]
- $fo(goh)(x) = (fog)oh(x)$ [Associative]

$f = \{(1,2)(3,2)(4,5)\}$

$g = \{(3,1)(2,5)\}$

gof ~~fog~~ $= \{(1,5)(3,5)\}$ start with f

fog ~~gof~~ $= \{(3,2)\}$ Start with g

1) $foI = Iof = f$ $f: A \to A$ on $A \to B$ $[foI_A = I_B of = f_{A\to B}]$

2) for $f\ A \to A$ which is a bijection

$fof^{-1} = f^{-1}of = I_A$

3) for $f\ A \to B$ which is a bijection

$$\begin{bmatrix} fof^{-1} = I_B \\ f^{-1}of = I_A \end{bmatrix}$$

$f = \{(1,a)(2,b)(3,c)\}$

$f^{-1} = \{(a,1)(b,2)(3,c)\}$

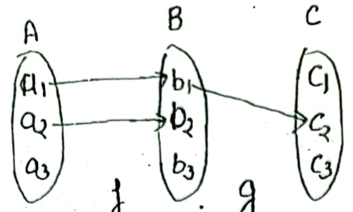
$fof^{-1} = \{(a,a)(b,b)(c,c)\}$
$= I_B$

$f^{-1}of = \{(1,1)(2,2)(3,3)\}$
$= I_A$

<u>Note</u> 1. If f & g are one to one $\Rightarrow gof$ is one to one

2) f & g are onto $\Rightarrow gof$ is onto

3) f & g are one-one & onto $\Rightarrow gof$ is bijection



gof is one to one
fog is also one-to one.

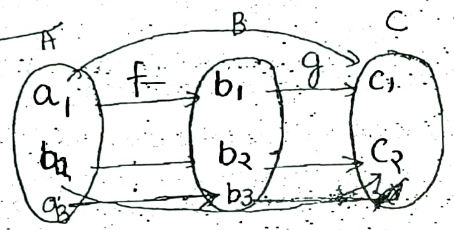
g: B→C
f: A→B

~~it doesnot exist here~~

..... Same happens with onto.

<u>ues</u>- if fog is onto than both f & g are onto
False

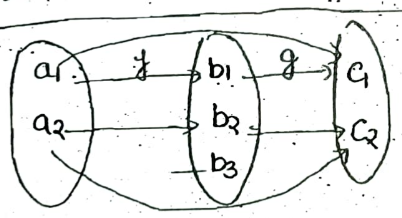
if fog is onto than atleast one of them is onto
True



g: B→C
f: A→B
gof: A→C

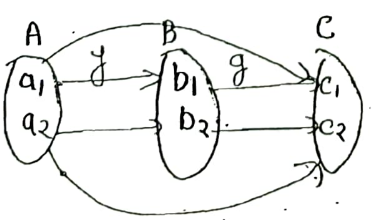
$g \circ f = \{(b_1, c_1)(b_2, c_2)\}$
[A→C]
↳ onto

Here g need not be onto
f need to be



$g o f(x) = g(f(x))$
$= \{(a_1, c_1)(a_2, c_2)\}$
↓ ~~one to one~~ onto

f is not onto
g is onto



$f = \{(a_1, b_1)(a_2, b_2)\}$
$g = \{(b_1, c_1)(b_2, c_2)(b_3, c_1)\}$
$g o f = \{(a_1, c_1)(a_2, c_2)\}$
↓ one to one

g need not to be one to one
f need to be one to one

GROUP THEORY

*1. Binary operation & Alzebraic Structure & properties.
2. Properties of Abelian group
*3. Classic Examples of group
*4. Properties of group
*5. Power of an element of a group
6 Order of a group [cardinality of group]
*7. Order of an element of group
*8> cyclic group
9> Subgroup
10> Lagrange's Theorem

- Cryptography
- Automatic Error correction (AEC)

Every Alzebraic Structure $(S, *)$ can be classified into
- Groupoid
- Semigroup
- monoid
- group
- Abelian group

Binary operation –

$(S, *)$ $*$ is a binary operation iff $\boxed{a*b \text{ is unique.}}$
- operate on two things
- it should be a function

Which of the following is not Binary operation?

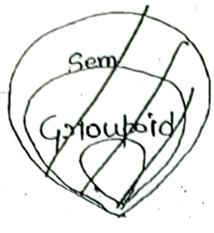
1. $a*b = a^2+b^2$
2. $a*b = a$
3. $a*b = a-b$
4. $a*b = \sqrt{ab}$ (because of being not unique)
5. $a* = \sqrt{a}$ - not a binary operation as well as not unique
6. $a*b = a$ or b [Not a binary operation]

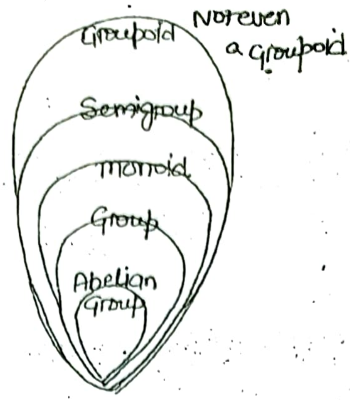
Note- Binary operation need not be closed.

Classification of a Alzebraic structure $(S, *)$

1. Groupoid — CLOSURE
2. SemiGroup. — CLOSURE + Associative
3. Monoid — closure + Associative + identity
4. Group — Closure + Associative + identity + inverse
5. ABelian Group — Group + commutative

Not even a groupoid - not closed





$N = \{0, 1, 2, \ldots\}$
$Z^+ = \{1, 2, 3 \ldots\}$
$Z = \{\ldots -2, -1, 0, 1, 2 \ldots\}$
$= Z^- \cup N$

Closure – In $(S, *)$
$\forall\, a, b \in S \quad a*b \in S$

Associative $\forall\, a, b, c \in S \quad a*(b*c) = (a*b)*c$

Identity $\exists e\ \forall a \quad a*e = a = e*a$ • identity is unique for all $a \in S$

Inverse $\forall a\ \exists a^{-1} \ni a*a^{-1} = a^{-1}*a = e$
• inverse is unique for $a \in S$

Commutative $\forall\, a, b \in S \quad (a*b) = (b*a)$

1. $(Z, *)$

	CLOSURE	ASSOCIATIVE	Identity	Inverse	Commutative	
$a*b = a+b$	✓	✓	✓ [0]	✓	✓	Abelian Group
$a*b = a \times b$	✓	✓	✓ [1]	✗	✓	Monoid but not group
$a*b = a^b$	✗ $a=2$, $b=-1$	$(a^b)^c = a^{b^c}$ ✗	✗	✗	✗	Not even a groupoid
$a*b = \max(a,b)$	✓	✓	✗	✗	✓	Semigroup but not monoid
$a*b = a$	✓	✓	✗	✗	✗	Semigroup but not monoid
$a*b = a+b-ab$	✓	✓	✓ [0]	✗	✓	Monoid but not group
$a*b = a^2+b^2$	✓	✗	✗	✗	✓	Groupoid not Semigroup
(table below)	✓	✓	✓ [a]	✓	✓	Abelian Group

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

- 1. In table, if all the element $\in S$ then it satisfies the closure

Associative $(a*b)*c = a*(b*c)$

for $a*b = a^b$

$(a^b)^c = a^{b^c}$

$a^{bc} \neq a^{b^c}$ (False)

- for $a*b = a+b-ab$

$a*(b*c) = (a*b)*c$

$a*(b+c-bc) = (a+b-ab)*c$

$a+b+c-bc-ab-ac+abc = a+b+c-ab-ac-bc+abc$

[True]

- for $a*b = a$

$a*(b*c) = (a*b)*c$

$a*b = a*c$

$a = a$ [True]

- for $a*b = a^2+b^2$

$a*(b*c) = (a*b)*c$

$a^2+(b^2+c^2)^2 \neq (a^2+b^2)^2+c^2$ [False]

Identity-

- for $a*b = a+b$ $\quad a+e = a = e+a \Rightarrow e=0$
- for $a*b = a*b$ $\quad a*e = a = e*a \Rightarrow e=1$
- for $a*b = a^b$ $\quad a*e = a = e*a \Rightarrow a^e = a = e^a$

$\Rightarrow$ Not possible

- for $a*b = \max(a,a)$ $\quad a*e = a = e*a$

$\max(a,e) = a = \max(e,a)$ $\quad [e = -\infty]$

$\hookrightarrow$ not defined uniquely

- for $a*b = a$

$a*e = e*a = a$

$a = e = a \Rightarrow$ does not exist (not defined uniquely)

- for $a*b = a+b-ab$ $\quad a*e = a = e*a$

$a+e-ae = a$

$e+\cancel{a}-ae = \cancel{a}$

$e = ae$

$e-ae$

$e(1-a) = 0$

$\boxed{e=0 \text{ or } a=1}$ $\quad \Rightarrow$ identity exist

$\therefore e=0$ for all $a \in S$

for $a*b = a^2+b^2$

$a*e = a = e*a$

$a^2+e^2 = a = e^2+a^2$

$e^2 = a-a^2$

$e^2 = a(1-a)$

$e = \pm\sqrt{a(1-a)}$

↳ does not exist [not unique]

<u>for table-</u>

∴ there should exist

*	a	b	c	d
a	a	b	c	d
b	b			
c	c			
d	d			

$\Rightarrow a*e = a$
$\Downarrow$
a

$e*a = a$
$\Downarrow$
a

<u>Inverse-</u>

- for $a*b = a+b$

$a*a^{-1} = e = a^{-1}*a$

$a+a^{-1} = 0 = a^{-1}+a$

$\boxed{a^{-1} = -a}$ ⇒ inverse exist

- for $a*b = a*b$

$a*a^{-1} = e = a^{-1}*a$

$a\times a^{-1} = 1 = a^{-1}*a$

$[a^{-1} = 1/a]$ if $a=2$ $a^{-1} = \frac{1}{2} \notin Z$

⇒ inverse doesnot exist

- for $a*b = a^b$

$a*a^{-1} = e = a^{*-1}$

→ doesnot exist.

- for $a*b = a+b-ab$

$a*a^{-1} = e = 0$

$a+a^{-1}-aa^{-1} = 0$

$a a^{-1} = a(1-a^{-1})$

$a^{-1} = \frac{a}{a-1}$ ⇒ does not exist $[a=1]$

- <u>for table-</u> Yes inverse exist bcoz in every row & every column 'a' exist. & a should be present at every mirror image position.

<u>Ques-</u> $a*b = a+b-ab$, one which set it will form a group.

1) R
2) Z
3) Z − {1}
4) ✓ R − {1}

Commutative - **for table**: if Table = transpose (Table)

i.e matrix should be a symmetric matrix.

Properties of an Abelian Group:

1. A Group G $(G,*)$ is Abelian if and only if $a*b = b*a$. $\forall\, a,b \in G$

2. A Group G is Abelian if and only if $(a*b)^2 = a^2 * b^2$

pf $(a*b)^2 \equiv (a*b)*(b*a)$ Since

$\equiv a*b*b*a$

$= a*b^2*a$

$= a*a*b^2$

$= a^2*b^2$.

$$
\left[
\begin{aligned}
(a*b)*(a*b) &\equiv (a*a)*(b*b) \\
((a*(b*a)*b)) &\equiv (a*(a*b)*b) \quad \text{[Associative]} \\
((a*(a*b)*b)) &\equiv (a*(b*a)*b) \quad \text{[commutative]} \\
a*a*b*b &\equiv (a*b)*(a*b) \\
a^2*b^2 &\equiv (a*b)^2 \quad \text{[Hence proved]}
\end{aligned}
\right.
$$

$$
\left[
\begin{aligned}
&G \text{ is abelian} \iff a*b = b*a \\
&G \text{ is abelian} \iff (a*b)^2 = a^2 * b^2
\end{aligned}
\right]
$$

3) If G is Group $(G,*)$ $\forall\, a \in G$ $a^{-1} = a \Rightarrow (G,*)$ is abelian.

$\forall\, a \in G$ $a^2 = e \Rightarrow (G,*)$ is abelian.

$a*b$
$a^{-1}*b^{-1}$
$= b^{-1}*a^{-1}$
$(b*a)$

$a*b$
$a^{-1}*b^{-1}$
$(b*a)^{-1}$
$a*a^{-1}$
$a^{-1}*a$

Ques- If $(G,*)$ is an Abelian group which of the following is true?

(a) $\forall\, a \in G$ $a^{-1} = a$

(b) $\forall\, a \in G$ $a^2 = e$

(c) ✓ $\forall\, a,b \in G$ $(a \cdot b)^2 = a^2 \cdot b^2$

(d) $(G,*)$ is finite group

lassic Example of Groups

- $(\{0,1\}, \oplus)$
- $(\{0,1,2,\ldots m-1\}, +_m) = (Z_m, +_m)$ — addition modulo m

  $Z_m = \{0,1,\ldots m-1\}$

  $5 +_3 7 \Rightarrow 5+7 \bmod 3$
  $\Rightarrow 12 \bmod 3$
  0

- $(\{1,2,3,\ldots p-1\}, \times_p)$ — multiplication Modulo p [p: prime No]
- $(S_n, \circ)$ — composition

  $S_n \to$ set of all permutation function on a set of n elements

$A = \{1,2,3\}$

$$S = \left\{ \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \right\}$$

- $(\{0,1\}, \oplus)$

$\oplus$	0	1
0	0	1
1	1	0

- it is closed
- XOR is associative
- identity element = 0
- inverse = $[a = a^{-1}]$ $\begin{bmatrix}0^{-1}=0\\1^{-1}=1\end{bmatrix}$
- commutative.

- Hence it is abelian group.
- used in Automatic error correction.

- $(\{0,1,2,\ldots m-1\}, +_m) \equiv (Z_m, +_m)$ $\boxed{\text{order} = m}$

$\{0,1,2,3\}$ $(Z_4, +_4)$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

- it is closed
- it is associative

as $(a +_m b) +_m c = a +_m (b +_m c)$

$[(a+b) \bmod m] +_m c \equiv a +_m ((b+c) \bmod m)$

$((a+b) \bmod m + c) \bmod m$

$= (a + (b+c) \bmod m) \bmod m$

$(a+b+c) \bmod m \equiv a+b+c \bmod m$

- Identity element $e = 0$
- inverse element

$$\begin{bmatrix} \boxed{a^{-1} = m-a} & a \neq 0 \\ a^{-1} = 0 & a = 0 \end{bmatrix}$$

- it is also commutative

$a +_m b \equiv b +_m a$

$\boxed{(a+b) \bmod m = (b+a) \bmod m}$

1. It is an abelian group
2. $\boxed{a^{-1} = (m-a) \bmod m}$
3. Used in cryptography
   [Ceaser-Cipher]

3. $(\{1, 2, 3, \dots p-1\}, \times_p) \equiv (Z_p, \times_p)$ = ~~{1, 2,~~

Let $Z_p = \{1, 2, 3, 4\}$

$Z_5 = \{1, 2, 3, 4\}$ $\boxed{\text{order} = p+1}$

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

- it is closed
- it is associative
- Identity element $e = 1$
- Inverse element-

$a \cdot a^{-1} \bmod p = 1 \Rightarrow a a^{-1} = px + 1$ $\quad a^{-1} = \frac{1}{a}(px+1)$

Ques- What is 2^{-1}, 3^{-1} in $(Z_{13}, \times_{13})$

$2 \cdot 2^{-1} \bmod 13 = 1$

$2^{-1} = 7$

$3 \cdot 3^{-1} \bmod 13 = 1$

$3^{-1} = 9$

- it is commutative

$a \times_p b = b \times_p a$

$a \cdot b \bmod p \equiv b a \bmod p$ [True]

- Hence it is an abelian group
- Used in RSA

Ques- $(\{1, 2, 3, 4, 5\}, \times_6)$ is not a group. (non prime)
Which of the following is not a reason?

a Closed → ~~true~~ False

$3 \times 2 \bmod 6 = 0 \notin S$ [Closure & inverse]

Associative - True

Identity → true

Inverse → does not exist $(2^{-1}, 3^{-1})$

Permutation Group $(S_n, \cdot)$:

$A = \{1, 2, 3\}$

$(S_3, \cdot)$ $S_3 = (f_1, f_2, f_3, f_4, f_5, f_6)$

- order of $S = 6$ $(n!)$
- Identity element $(f^n) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

- it is closed
- Inverse always exist

ex - $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{\text{inverse}} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

Ques what is its inverse?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 5 & 7 & 6 & 4 \end{pmatrix}^{-1}$$

Ans-
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 7 & 4 & 6 & 5 \end{pmatrix}$$

∴ we can also say inverse will exist for every ~~composition~~ function.

- it is also associative
- Hence it is also a group
- used in physics.
  used in cryptography.

- it is not commutative. hence not an abelian group

$$\left[\begin{array}{c} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \end{array}\right]$$

Hence it is not commutative

22-Sept-2017

Properties of Groups:

1. Identity element (e) is unique for a group
2. Left Identity should be same as right identity

$$a * e = e * a$$

same

3. a^{-1} is unique for a given element a.
4. $(a*b)^{-1} = b^{-1} * a^{-1}$
5. $(a^{-1})^{-1} = a \Rightarrow a^{-1} = b$ iff $b^{-1} = a$

ie $\boxed{ab = e = ba}$

Ex if G is group which is true

1) $(a*b)^{-1} = a^{-1} * b^{-1}$

2. $(a*b)^{-1} = b^{-1} * a^{-1}$ [2 is true]

if G is Abelian group than both are true

6. Left Cancellation & Right Cancellation hold for a group but not monoid.

$a*b = a*c \Leftrightarrow b = c$

$b*a = c*a \Leftrightarrow b = c$

Pf- $a*b = a*c$

multiply by a^{-1} on both sides,

$a^{-1}*(a*b) = a^{-1}*(a*c)$

$\boxed{b = c}$

for Monoid

$b = c \Rightarrow ab = ac$

$b = c \Rightarrow ba = ca$

Ques- is it true? $a*b = b*a$

1) for a group → No bcz group need not be commutative

2) for an abelian group

→ True because it is commutative

7. The Equ^n $ax = b$ and $ya = b$ has unique solution for x, y.

$\boxed{x = a^{-1}b \quad y = ba^{-1}}$

$ax = b \Rightarrow a^{-1}*(ax) = a^{-1}b$

$x = a^{-1}b$

↳ unique as a^{-1} is unique

$a^{-1}*b$ is also unique

> Binary operation Table of ~~tabt~~ a finite group (Cayley Table) does not allow any repitition in a row and in a column.

If repitition is allowed

	a	b	c
a	a	b	c
b	b	b	b
c	c	b	a

- $b*a = b$ $\Rightarrow$ $b x^{\circ} = b$
  $b*b = b$ $\quad$ ↳ x has two soln

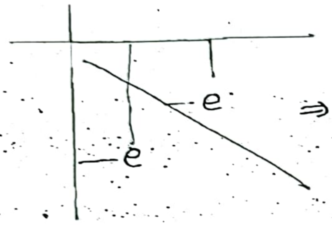
∴ it can never allow the repitition in a row.

$c*b = b$
$a*b = b$

$Yb = b$
↓ two soln

[No repitition in row]

> In a Cayley table, if for an off diagonal element the value is e then its mirror image should always be e.



$\Rightarrow$ due to $\boxed{a^{-1}*a = e = a*a^{-1}}$

Fill the blanks for given Cayley Table

using le = re
No repitition
$a^{-1}*a = e = a*a^{-1}$

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

No repitition

[using No repitition]

using Mirror Image

due to left Identity = right identity

Powers of an element of group:

$(G,*)$ $a \in G$ $\quad a^n \quad n \in \mathbb{Z}$ –

- $a^0 = e$ [Identity elmt]
- $a^1 = a$

Ex. for $(\mathbb{Z}, +)$ let $a = 2$

$2^0 = 0$

- $a^2 = a*a$
- $a^3 = a*a*a = a*a^2 = a^2*a$

- $a^{-1} = a^{-1}$
- $a^{-2} = a^{-1} * a^{-1}$
  $= (a^2)^{-1}$
- $(a^m)^n = a^{mn}$
- $a^m . a^n = a^{m+n}$
- $\left((a^3 a^{-1})^2 (b^3)^8 \right)^{-1}$
  $= ((a^2)^2 (b^{24}))^{-1}$
  $= b^{-24} * a^{-4}$
  $= (b^{-1})^{24} * (a^{-1})^4$
  $= d^{24} * a^4$
  $= d^{24 \bmod 4} * a^4$
  $= d^0 * a = a * a = \underline{a}$

$(Z, +)$

Here $e = 0$

$a^{-1} = -a$

$2^{-2} = 2^{-1} * 2^{-1}$
$= -2 + -2$
$= -4$

$2^{-1} = -2$

$2^0 = 0$

$2^1 = 2$

$2^2 = 2 * 2$
$= 2 + 2$
$= 4$

$2^3 = 2 * 2 * 2$
$= 4 + 2$
$= 6$

$2^n = 2n$

$(R^+, \times)$

$e = 1$

$a^{-1} = 1/a$

$2^{-2} = 2^{-1} * 2^{-1}$
$= 1/4$

$2^{-1} = 1/2$

$2^0 = 1$

$2^1 = 2$

$2^2 = 2 * 2$
$= 2 * 2$
$= 4$

$2^3 = 2 * 2 * 2$
$= 2 \times 2 \times 2$
$= 8$

$2^n = 2^n$

Note-
- $O(G) \geq O(a)$
- $a^n = a^{n \bmod (O(a))}$
- $e^n = e \quad [n \in Z]$
- Order of element ~~can~~ be infinity
- order of group can be infinity

Here $\left.\begin{aligned} O(a) &= 1 \\ O(b) &= 4 \\ O(c) &= 2 \\ O(d) &= 4 \end{aligned}\right\}$ order of element

[using previous Cayley Table]

Order of a Group-
- $O(G)$
- it denotes the cardinality of group

$\{(0,1) \oplus\} \Rightarrow O(G) = 2$

- Every permutation function is either even or odd.
- $(E_p, \cdot) \Rightarrow$ Even permutation group $O(G) = \frac{n!}{2}$
- $(S_n, \cdot)$ = Even permutation + odd permutation
- No of even permutation = No of odd permutation
- Identity permutation is always even.

Odd permutation never form a group

$(O_n, \circ) \Rightarrow$ Not a group

- Not closed $\quad O_n \cdot O_n = E_n$
- Not have Identity $(e = E_n)$
- ~~does no~~ inverse doesn't exist.
- but it is associative

$(E_n, \circ) \Rightarrow$ group

- Closed $\quad E_n \cdot E_n = E_n$
- Associative
- Identity $e = E_n$
- inverse exist.

- it is also known as Alternating permutation group of n letters
- used to solve topological puzzles.

<u>Even permutation</u>- A permutation is even if it can be ~~pro~~ broken down in even No of transpositions.

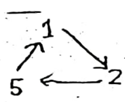
Every permutation which is not a cycle can be broken down into product of disjoint cycles.

Every permutation [which is not a transposition] can be broken down into product of transposition [No of transposition will be unique]

~~odd transposit~~

<u>Odd permutation</u>- if No of transposition is odd.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 3 & 4 & 1 & 6 & 7 \end{pmatrix}$$



To break it into cycle-

- Start with 1 & reach again to 1.
- do it for remaining elmt.
- leave the identity element.

$= (1, 2, 5)$

↳ one cycle exist

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 4 & 3 & 1 & 6 & 7 \end{pmatrix}$$

$= (1\ 2\ 5), (3\ 4)$ → product of two cycle permutation

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 3 & 4 & 1 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 3 & 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 4 & 3 & 1 & 6 & 7 \end{pmatrix}$$

Transposition- It is a cycle of length 2

- To break a cycle in ~~composition~~ transposition-
  - $(1, 2, 5)$
  - $(1,5)(1,2)$

— • Transposⁿ for a permutation will not be unique

Ex- $(3, 1, 2, 5)$

- $(3,5).(3,2).(3,1) \neq (3,2)(3,5)(3,1)$

↳ order matters here

Ex- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 3 & 4 & 1 & 6 & 7 \end{pmatrix}$

$= (1,2,5)$

$= (1,5)(1,2) \rightarrow$ Even permutation

• break down of a cycle into transposition will be unique.

2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 4 & 3 & 1 & 6 & 7 \end{pmatrix}$

$(1,2,5).(3,4) \equiv (3,4)(1,2,5)$

$(1,5)(1,2)(3,4) \equiv (3,4)(1,5)(1,2)$ [Two transposition]

↳ odd permutation

3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$

$= (\ \ )$

= Even permutation [Identity per. is even]

Note-

$$\boxed{\begin{aligned} O \cdot O &= E \\ E \cdot E &= E \\ O \cdot E &= O \\ E \cdot O &= O \end{aligned}}$$

[odd no of transposⁿ + odd no of transⁿ = even no of transposⁿ]

Order of an element of group $O(a)$:

In $(G, *)$ $\forall a \in G$ ~~O(a)~~

Properties-

1. ~~O(a)~~ $\leqslant O(G)$
2. $O(a)$ divides $O(G)$ i.e $O(a) \div O(G)$ exist [for finite group]
3. $O(a)$ is the smallest positive integer such that $a^n = e$

$$\boxed{O(a) = n \text{ s.t } a^n = e \quad n \in Z^+}$$

- if no such integer exist than
  $O(a) = \infty$ [in case of infinite group]
- for a finite group, $O(a)$ will exist & surely be finite.

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

~~O(a)~~

Here $e = a$

Note- order of identity element is always 1.

Ques- which is false?

1) there exist 2 elmt in a group with order 1. False
2) there exist 1 elmt in group with order 1. True

$O(a) = 1$

$O(b) = b' = b \quad = 4$
$b \cdot b = c$
$b^3 = c \cdot b$
$= d$
$b^4 = d \cdot b = a$

$O(c) \Rightarrow c \cdot c = a$
$\therefore O(c) = 4$

$O(d) \Rightarrow d' = d$
$d^2 = c$
$d^3 = b$
$d^4 = a$
$\Rightarrow O(d) = 4$

4) $O(a) = O(a^{-1})$

$\therefore$ if $O(a) = n \Rightarrow a^n = e$
$\Rightarrow (a^n)^{-1} = e^{-1}$
$a^{-n} = e \Rightarrow (a^{-1})^n = e \Rightarrow \underline{O(a^{-1}) = n}$

Ques- for a group of order 8 which of the following is not order of elmt in that group?

1) 1 2) 2 3) 4 4) 6 ✓

O(a) should divide O(G)

Quese For a group of order 8, which is false?

(a) There is no elmt with order 6.

(b) There is atleast one elmt with order 4. ✓

(c) There is atleast one elmt with order 1.

Note- • In a group with order ~~p~~ to be prime,

No of elements whose order is $p = p-1$

• In every infinite group, there exist an element with finite order

Ex- • $(Z, +) \to$ $O(0) = 1$, $O(-1) = \infty$

$O(1) = \infty$, $O(-2) = \infty$

$O(2) = \infty$, $O(-3) = \infty$

$O(3) = \infty$

$(Z, -)$ → not even a group

$O(0) = 1$

$O(1) = 2$

$O(2) = 2$

$O(1^{-1}) = 2$

Property 5) $O(a*b) = O(b*a)$

$O(a*b) = n$

$(a*b)^n = e$

let $a*b = c$ than in a group $c^n = e$

If $a*b = c$, due to mirror image of lower triangle & upper triangle

$b*a = c \Rightarrow c^n = e$

$\therefore O(a*b) = O(b*a)$. } for abelian group

6) $O(x^{-1}*a*x) = O(a)$

$O(x*a*x) = O(a)$

• Let $O(a) = t$

$\therefore a^t = e$

$(x^{-1}*a*x)^t = (x^{-1}*a*\cancel{x})*(\cancel{x^{-1}}*a*\cancel{x})*(\cancel{x^{-1}}*a*x)\ldots *(x^{-1}*a*x)$

$= x^{-1}*a^t*x$

$= x^{-1}*e*x = x^{-1}*x = \underline{e}$

ote- 1. $(M_{\substack{non \\ singular}}, X)$ – form a group $\quad M_{\substack{non \\ singular}} * M_{\substack{non \\ singular}} = M_{\substack{non \\ singular}}$ of same order $[n \times n]$

2. $(M_{\substack{square \\ Matrix}}, X) \rightarrow$ Monoid [Inverse may or may not exist]

roperty 7- $a^n = a^{n \bmod (O(a))}$

8) If $O(a) = n \Rightarrow a^m = e$ iff $m = nk \quad k \in Z$

Ques- if $O(a) = 5$ what can you say about a^{27}?

$[a^{27} \neq e]$

$a^{27} = a^2$

clic group- A Group $(G, *)$ is cyclic iff

$\exists g \in G \quad \forall a \in G \quad \boxed{a = g^n} \quad n \in Z$

- multiple g can exist.

1. $(Z, +)$ is a cyclic group.

Here g is known as generator of the group.

Here $g = 1, -1$

$[\ \ldots -2 = 1^{-2}, -1 = 1^{-1}, 0 = 1^0 \quad 1 = 1^1 \quad 2 = 1^2 \quad 3 = 1^3 \ldots\ ]$

$[\ \ldots (-2) = (-1)^2, -1 = -1^1, 0 = -1^0, 1 = -1^{-1}, 2 = -1^{-2} \ldots\ ]$

lote- • identity element can never the generator of group. ~~exect~~ except trivial group

$[\{e\}, *]$

*	e
e	e

ote- 1. If g is generator for a group $(G, *)$ than g^{-1} is also a generator.

$a = g^n$

$= (g^{-1})^{-n}$

Property 2- 1. ~~If~~ A cyclic group is always an abelian group

$$a*b = b*a$$
$$g^n * g^m = g^m * g^n$$
$$g^{n+m} = g^{m+n}$$

Hence $a*b = b*a \Rightarrow$ it will always be abeliangroup

- An Abelian group need not be cyclic.
  Ex- $(R, +)$ (all elements can not be generated)
- A non-abelian group will always be non-cyclic.

Standard Cyclic group

1. $(Z_m, +_m)$ $g = 1, m-1$

2 $\{(n, n^{th}$ root of unity$), \times\}$

$n=1$ $(\{1\}, \times)$ $\rightarrow g = 1$

$n=2$ $((1,-1), \times)$ $g=-1$

$n=3$ $(\{1, \omega, \omega^2\}, \times)$ $g = \omega, \omega^2$

$\omega^0 = 1$ $(\omega^2)^0 = 1$
$\omega^1 = \omega$ $(\omega^2)^1 = \omega^2$
$\omega^2 = \omega^2$ $(\omega^2)^2 = \omega^3 \omega = \omega$

$n=4$ $(\{1,-1,i,-i\}, \times)$ $g = \underbrace{i, -i}_{\Downarrow}$

both are inverse of each other

$i^0 = 1$ $(-i)^0 = 1$
$i^2 = -1$ $(-i)^1 = -i$
$i^1 = i$ $(-i)^2 = -1$
$i^3 = -i$ $(-i)^3 = i$

check whether it is ~~gr~~ cyclic group

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$\Rightarrow$ it is cyclic & isomorphic to $(Z_4, +_4)$ (One shifted)

What are generator? (b, d)

- Start with b $b^0 = a$
  $b^1 = b$
  $b^2 = c$
  $b^3 = d$

it is a cyclic group

$\therefore$ b is a generator.

$b^{-1} = d$ Hence it is also a generator.

$c = c^0 = a$
$c^1 = c$
$c^2 = a \Rightarrow$ Not a generater

* $(R, +)$ ⇒ It is a group.
⇒ not a cyclic group.
⇒ no generator exist.

roperties -

1) Every prime order group is always cyclic.
pf- there surely exist an elmt with order p
then it will always be cyclic
bcoz the element with order p will always be the generator

∴ $\boxed{\text{No of generator} = p-1}$
(other than identity element)
p

* If in a (finite) group an elmt exist such that $O(a) = O(G)$ then surely G is a cyclic group.

ues- for a group$(G, *)$ with order 7 $\{a, b, c, d, e, f, g\}$ with identity element = a
What is the generator of G?
generators = $\{b, c, d, e, f, g\}$

operty2 - A finite group $(G, *)$ is cyclic group if and only if there exist $a \in G$ s.t $O(a) = O(G)$.
pf:- Let n be the no of elmt
if $O(G) = n$ & $O(a) = n$, then

$$\begin{cases} a^0 = e \\ a^1 \\ a^2 \\ \vdots \\ a^n = e \end{cases}$$

till [0 to n-1] → it has generated the distinct element

∴ No of distinct elmt generated = n
⇒ every elmt is generated
than, a will be the generator of group

Property 3- For a cyclic group with order n i.e $O(G)=n$,
then No of generator of that group $= \phi(n)$ [Euler's Totient Function]

Euler's Totient f^n- if n is the prime No

$\phi(n) = n-1$

else

if $n = P_1^{n_1} * P_2^{n_2} * P_3^{n_3} \ldots$

$\therefore \phi(n) = \phi(P_1^{n_1} * P_2^{n_2} * P_3^{n_3}) \ldots$

$= \phi(P_1^{n_1}) * \phi(P_2^{n_2}) * \phi(P_3^{n_3}) \ldots$

$= (P_1^{n_1} - P_1^{n_1-1})(P_2^{n_2} - P_2^{n_2-1})(P_3^{n_3} - P_3^{n_3-1})$

$\phi(n) \Rightarrow$ ~~No o~~ ~~no of~~ count of no's from 1 to n such that $\gcd(x, n) = 1$

Ex- $n = 60$

$n = 2^3 * 3 \times 5$

$\therefore \phi(n) = (2^3 - 2^2)(3^1 - 3^0)(5^1 - 5^0)$

$= 2 * 2 * 4 = 16$

No of generator of a group with order n $= \phi(n)$ bcoz $\phi(n)$ represent no which are relatively prime to n from 1 to n bcoz if they are relatively prime than there order will be n.

Note- If $g_1, g_2, g_3, g_4, \ldots g_k$ be the generator of group with order n
then each $g_1, g_2, g_3, \ldots g_k$ can be written in form of g_1^x

g_2^x

and so on.

this x will also be relatively prime to n for all the generators

Q1 $G = (\{1, 2, 3, 4, 5, 6, 7, 8\}, \times_8)$

No of Generators, $\phi(8) = 2^3 - 2^2$
$= 4$

$\phi(8) = \{1, 3, 5, 7\}$ [Relatively prime to n]

$1 \; (1)^3 \; (1)^5 \; (1)^7$

Q

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

No of generator = 2
(b, d)

$\begin{bmatrix} b \to b^1, b^3 \\ d \to d^1, d^3 \end{bmatrix}$ these power also should be relatively prime to n

Ques- for a group with order = 60 which of the following will not be generator? if g is a generator

(a) g^{13}

(b) g^{17}

(c) g^{23}

(d) g^{12} ✓

[g^{12} will not be generator bcoz $\gcd(12, 60) \neq 1$]

$(\{a, a^3, a^5, a^7\}, \times)$ - cyclic group everything can be written as power of a

$(\{a^2, a^3, a^5, a^7\}, \times) \to$ Not a cyclic group

Subgroup A $(H,*)$ is subgroup of $(G,\times)$ iff
1. $H \subseteq G$
2. $(H,*)$ itself form a group.

Which of the following is not a subgroup of $(Z_4, +_4)$

a) $(\{0,1,2\}, +_4)$ — $1^{-1} = 3 \notin \{0,1,2\}$ Hence Not a subgroup
- It is not closed.

b) $(\{0,2\}, +_4)$
c) $(\{0\}, +_4)$
d) $(\{0,1,2,3\}, +_4)$

Note- • $(\{e\}, *)$ is trivial subgroup of $(G,*)$
• $(\{G\}, *)$ ——— of $(G,*)$

Every [non-trivial] Group has exactly two trivial subgroup

• A trivial Group has only one trivial subgroup.
$(\{e\}, *)$

For $(Z_4, +_4)$
- There is no group with 0 element
1 element group - $(\{e\}, *)$
2 element group - $\{0,1\}$ $\{0,2\}$ $\{0,3\}$
Not a group

3 element group - not possible bcoz According to Lagrange's theorem
$O(H) \div O(G)$

4 elmt group - $(\{0,1,2,3\}, *)$

$\therefore$ No of possible subgroup = 3

Note- • A group with prime order will always have 2 subgroups
$(\{e\}, G)$

• [No of proper (non-trivial) subgroups for a group with prime order = 0]

Ques.- Which of the following is not a subgroup of $(Z, +)$

a) $(E, +)$

(b) $(0, +)$

(c) $(3Z, +)$

(d) $(-5Z, +)$

A- (b) $(0, +)$ → not closed, no identity, no inverse

Note- Any $(kZ, +)$ will always be subgroup of $(Z, +)$.

- No of subgroups of $(Z, +)$ = infinite.
- An infinite group can have infinite no of subgroup.
- A finite group will always have finite no of subgroup.

Lagrange's Theorem:

if $(H, *)$ is subgroup of $(G, *)$, than

$O(H) \div O(G)$

if $(H, *)$ & $(K, *)$ is subgroup of $(G, *)$ than

- $(H \cap K, *)$ is always a subgroup.
- $(H \cup K, *)$ may or may not be a subgroup. [closure may get violated]

- Infinite intersection of a subgroup will also be a subgroup

- for subgroups $(H, *)$ $(K, *)$ of $(G, *)$

$(HK, *)$ will always be a subgroup.

$HK = \{ h * k \mid h \in H, k \in K \}$

Formulae-
$$\boxed{O(HK) = \frac{O(H) \times O(K)}{O(H \cap K)}}$$

Ques- if H is a subgroup of G. which is false? $O(G) = 8$

1) a subgroup (atleast) exist with order 1

2) atleast a subgroup exist with order 4 ✓

2. $O(H \cap K) \lneq O(H)$

3. $O(H \cap K) \leq O(K)$

4. $O(H \cap K) \div O(H)$

5. $O(H \cap K) \div O(K)$

6. $O(HK) \div O(H)$ & $O(K)$

$(H, *)$ is a subgroup of $(G, *)$

(a) H is subset of G

(b) $\forall a, b \in H$

$a * b^{-1} \in H$

[closure, identity, inverse, associative]

Ques- if $O(H) = 7$, $O(K) = 3$ what is $O(H \cap K) = ?$

$O(H \cap K) = 1$

POSET, LATTICES & BOOLEAN ALGEBRA

1. POSET, TOSET, WOSET
   - POSET ↓ [Partial order Set]
   - TOSET ↓ [Total order Set]
   - WOSET ↓ [Well order Set]
2. Topological Sorting [conversion of POSET into TOSET]
3. HASSE diagram [Minimal Digraph]
4. Extremal elements of posets
   - Maximal, Minimal, Greatest, Least, GLB, LUB, GUB, UB
5. Lattice, Semilattice
6. properties of Lattice
7. Types of Lattice [Bounded, commutative, Distributive]
8. Boolean Algebra

POSET, TOSET, WOSET:

POSET - partial order set
[Reflexive, Antisymmetric, Transitive]

- $(S, \leq)$ – partially ordered set
  ↳ non empty set

- $x \leq y$ → x is related to y with some partial order opⁿ/relation

1. $R = \{(x, y) \mid x \leq y\}$
   ↳ Partial order Relation

1. $(Z, \leq)$, $(R, \leq)$, $(Z, \geq)$, $(R, \geq)$
2. $(\{1, 10, 15, 20\}, \leq)$
3. $(S, \subseteq)$ S - Set of all possible set, $(S, \supseteq)$
4. $(P(S), \subseteq)$
5. $(Z^+, /)$ $(Z^+$, x is divisible by y$)$
6. $(Z^-, /)$

6. $(D_n\ /)$ D_n - Set of all (+)ve Integeral Divisors of n.

$D_8 = \{1, 2, 4, 8\}$

$D_{10} = \{1, 2, 5, 10\}$

7. ($R \times R$, product partial order) $(x_1, y_1) R (x_2, y_2)$ iff $x_1 \leq x_2$, $y_1 \leq y_2$

8 ($Z \times Z$, product partial order)

1) $(Z, /)$ → Not a poset

for 0 → Not closed

$(2, -2)$ $(-2, 2)$ → Not antisymmetric

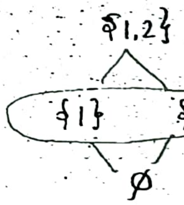
TOSET - • totally ordered set

• It will be a POSET

• Each & every element can be putted in an order

Ex- $S = \{1, 2\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$



→ Not comparable elements.

• POSET can contain non comparable elements.

but in TOSET, it is not allowed.

A $(R; \leq)$ will be TOSET iff

every pair $(x, y) \in R$ should be either $x \leq y$ or $y \leq x$

1. $(S, \subseteq)$ - Not a toset

2. $(P(S), \subseteq)$ - Not a toset

Note - • the Hasse diagram of a Toset is always a chain.

• A Toset need not be finite.

• the hasse diagram may or may not be finite.

• A poset is a Toset if its hasse diagram is a chain.

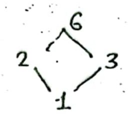
• Toset is also known as chain or linearly ordered set.

3. $(Z, \leq)$ $(R, \leq)$ → Always a Toset

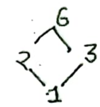
4. $(\{1, 10, 15, 6\}, \leq)$ → Always a Toset

5. $(Z^+, /)$ → Not a toset

$(Z^-, /)$ → Not a toset



6. $(D_n, /)$ → Not a toset



$D_6 = \{1, 2, 3, 6\}$

7. $(R \times R$, product partial order$)$ → Not a toset. $(1, 2)$ $(2, 1)$

8. $(Z \times Z, PPO)$ → Not a toset

<u>Ques</u>- $(P(S), \subseteq)$ is a TOSET iff

[S contains 0 or 1 element
$|S| = 0$ or 1]

9. $(R \times R, PPO)$ $(x_1, y_1) R (x_2, y_2) \Rightarrow x_1 < x_2$
$y_1 < y_2$

↳ <u>Not a toset</u>

- $(D_p, /)$ p - prime No
  ↳ Always a toset
- $(D_{2^n}, /)$ $n \in Z$
  ↳ Always a Toset
- $(D_{k^n}, /)$ $k \in Z$ $n \in Z^-$
  ↳ Always a Toset

<u>WOSET</u>- · Well ordered Set

- A $(S, \leq)$ is woset
  ⇒ if TOSET
  +
  has <u>least element</u>
  +
  set is discrete in nature [CI or CF not UCI]
- WOSET is comparable in nature

1. $(Z, \leq)$ → Not a woset as no least elmt. exist

$(Z^+, \leq)$ → WOSET

$(R^+, \leq)$ → Not a woset No Least elmt ('+ does not involve 0)

$(R^+ \cup \{0\}, \leq)$ → Not a woset
- Least element exist
- but, set is not discrete.

$(\{1, 2, 5, 10\}, \leq)$ - WOSET

<u>Note</u>- A Finite Toset is always a WOSET

$(Z^+, \geq)$ - Not a woset
↓
No least element exist

$(Z^-, \leq)$ → Not a woset No Least elmt exist $\overset{x}{(-2},\overset{y}{-1)}\,(-3,-2)$

$(Z^-, \geq)$ → WOSET $(-1, -2)\ (-2, -3)$

-1
-2
-3

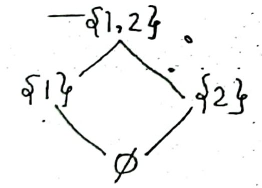
-3
-2
-1

$\boxed{\text{complex No will be POSET but not toset}}$ [because elements with or without i cant be compared]

Topological Sorting

for $S = \{1, 2\}$

$(P(S), \leq) = (\{\phi, \{1\}, \{2\}, \{1,2\}\}, \subseteq)$



- A Toset which is compatible with given poset

Topological Sorts:

$\phi \leq \{1\} \leq \{2\} \leq \{1,2\}$

$\phi \leq \{2\} \leq \{1\} \leq \{1,2\}$

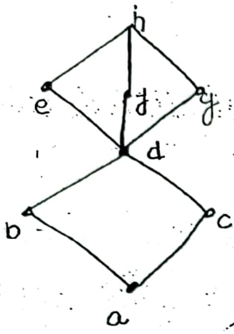
Not a topo. Sort

$\phi \leq \{2\} \leq \{1,2\} \leq \{1\}$

- Topological sort may or may not be unique.

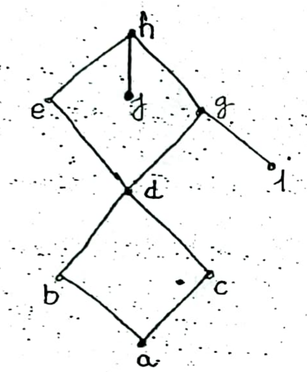
Algorithm- 1. Take minimal Element
2. Delete it
3. Repeat step 1.

How Many topological Sort exist in it?



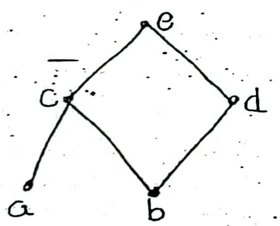
$$a \le \underbrace{\_ \le \_}_{\substack{2! \\ (bc,\ cb)}} \le d \le \underbrace{\_ \le \_ \le \_}_{\substack{3! \\ (e,f,g)}} \le h$$

$$\boxed{\text{No of topological Sort} = 2! \times 3! = 2 \times 6 = 12}$$



$a \le f \le i \le b \le c \le d \le g \le e \le h$

No of minimal elmt = 3

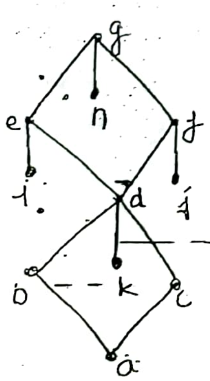


$a \le b \le c \le d \le e$

$b \le d \le a \le c \le e$

$a \le b \le c \le d \le e$
$a \le b \le d \le c \le e$
$b \le a \le d \le c \le e$
$b \le a \le c \le d \le e$
$b \le d \le a \le c \le e$

No of topological sort = 5

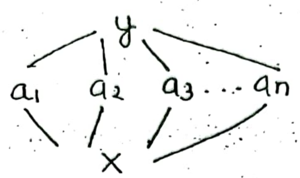


No of minimal element = 5

Ques - Consider a poset such that

$x \leq a_i \quad i = 1 \ldots n$

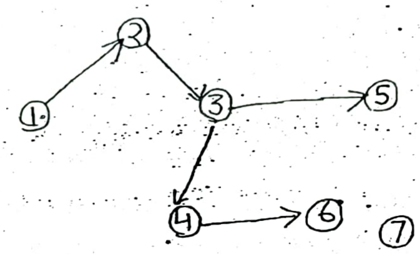
$a_i \leq y \quad i = 1 \ldots n$



How many toset which are compatible to poset?

$\boxed{\text{No of topological sort} = n!}$

- In case of digraph, minimal element is the element with zero indegree.



Topological Sort;

~~1 2 3~~

$\boxed{\text{No of minimal elmt} = ? \quad [1, 7]}$

$1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6 \leq 7$

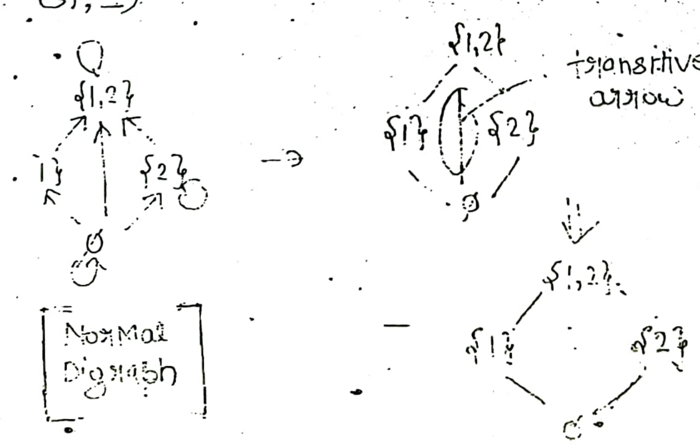
-Sept-2017

Hasse Diagram · minimal Diagraph

$S = \{1, 2\}$

$P(S) = \{\emptyset, \{1\} \{2\} \{1, 2\}\}$

$(P(S), \subseteq)$



Normal Digraph

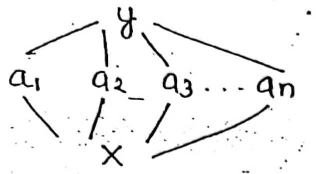
transitive arrow

- In hasse diagram, every element is related to itself.
- every element is related to elmt up to it, no need to draw arrow.
- No need of circle but the points directly.
- if there is any transitive arrow, remove it.
- a is related to b if there is an upward path between them

Ques- Consider a poset such that

$x \leq a_i \quad i = 1 \ldots n$

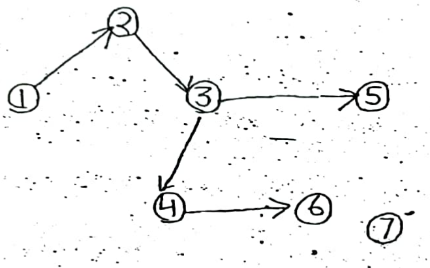
$a_i \leq y \quad i = 1 \ldots n$



How many toset which are compatible to poset?

$\boxed{\text{No of topological sort} = n!}$

- In case of digraph, minimal element is the element with zero indegree.



Topological Sort:

~~1 - 2 - 3~~

$1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6 \leq 7$

$\boxed{\text{No of minimal elmt} = 2 \quad [1, 7]}$

5-Sept-2017

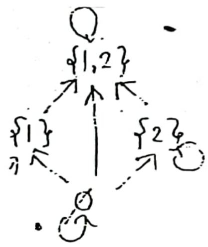
Hasse Diagram

- Minimal Digraph

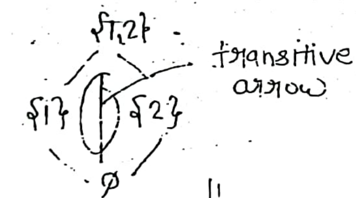
$S = \{1, 2\}$

$P(S) = \{\phi, \{1\} \{2\} \{1,2\}\}$

$P(S), \subseteq)$



Normal Digraph



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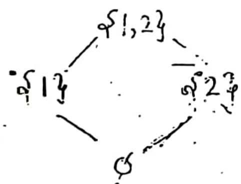
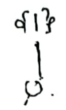


- In hasse diagram, every element is related to itself.
- every element is related to elmt up to it, no need to draw arrow
- No need of circle but the points directly.
- If there is any transitive arrow, remove it.
- a is related to b if there is an upward path between them

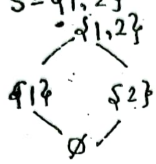
No of edges in a POSET :

1. $(P(S), \subseteq)$

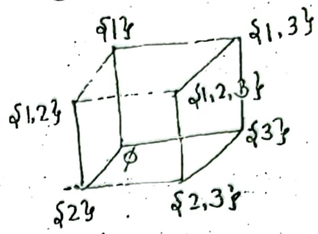
if $S = \{1\}$



If $S = \{1, 2\}$



$S = \{1, 2, 3\}$



∴ if set has n elements

$$\boxed{\begin{array}{l}\text{No of vertices} = 2^n \\ \text{No of edges} = n \cdot 2^{n-1}\end{array}}$$

[All are boolean Algebra]

if m is the size of P(S)

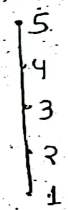
$$\boxed{\begin{array}{l}\text{No of vertices} = m \\ \text{No of edges} = \dfrac{m \log_2 m}{2}\end{array}}$$

2. for a toset with n elements

$\boxed{\text{No of edges} = n-1}$

No of vertices = n.

$\{1, 2, 3, 4, 5\}$



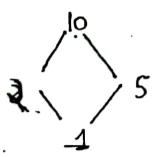
3. $(D_n, /)$ → No specific formula exist

4. $(D_p, /)$ → No of edges = 1

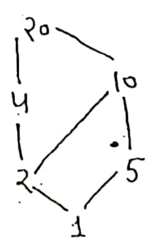
p-prime No

$(D_{10}, /) = (\{1, 2, 5, 10\}, /)$

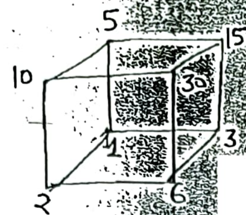
- Draw 1
- put all prime No & so on.



$(D_{20}, /) = (\{1, 2, 4, 5, 10, 20\}$



$(D_{30}, /) = (\{1, 2, 3, 5, 6, 10, 15, 30\}$



[No of edges = 12]

Note- A hasse diagram is boolean Algebra iff it is isomorphic to $(P(S), \subseteq)$

To check which $(D_n, /)$ is a boolean Algebra

$(D_n, /)$ is a boolean Algebra iff n can be broken down in distinct prime No. (no repetition of a prime No)

Ex $(D_{30}, /) \Rightarrow 30 = \{2, 3, 5\} \Rightarrow$ It is a boolean Algebra

$60 = \{2^2 \times 3 \times 5\} \Rightarrow$ Not a boolean Algebra

$(D_p, /) \rightarrow$ it is always a boolean Algebra (isomorphic to $P(S), \subseteq$) where $|S| = 1$

$(D_{210}, /) \Rightarrow 210 = 2 \times 3 \times 7 \times 5$

↳ Hence it is a boolean Algebra

$$\left[\begin{array}{l} \text{No of edges} = n \cdot 2^{n-1} \\ \text{No of vertices} = 2^n \end{array}\right]$$ where n = No of distinct factors

Note. $(D_n, /)$ is always a lattice

$(D_9, /) = (\{1, 3, 9\}, /)$

9
|
3 $\Rightarrow$ not isomorphic to $(P(S), \subseteq)$ $\Rightarrow$ Not a boolean Algebra
|
1

Which is not a boolean Algebra?

(a) $(D_{10}, /)$

(b) $(D_{33}, /)$

(c) $(D_{40}, /)$

(d) $(D_{20}, /)$

Ques. $(D_{210}, /)$ is a boolean Algebra to which $(P(S), \subseteq)$ it is isomorphic?

$210 = 2 \times 3 \times 5 \times 7$

No of factors $= (1+1)*(1+1)*(1+1)*(1+1)$
$= 16$

[Hence it will be isomorphic to $(P(S), \subseteq)$ where $|S| = 4$]

Note- if a no x can be written as

$$x = p_1^{n_1} * p_2^{n_2} * p_3^{n_3} * \ldots$$

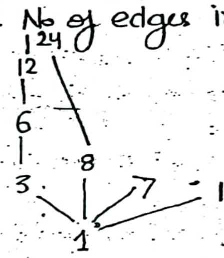
where $p_i \rightarrow$ prime No

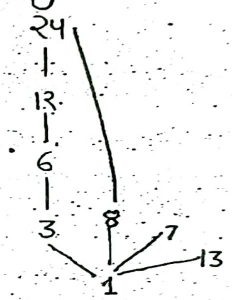
$n_i \rightarrow$ no of time P_i occurs

$$\boxed{\text{No of factors} = (n_1+1)(n_2+1)(n_3+1)\ldots}$$

Ques- $(\{1, 3, 6, 7, 8, 13, 12, 24\}, /)$

Find No of edges in the hasse diagram?



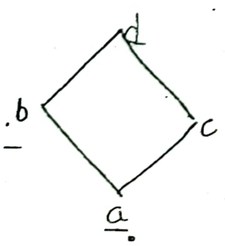


$\therefore$ $\boxed{\text{No of edges} = 8}$

Extremal Elements of a hasse diagram: for a poset $(S, \leq)$ $(Z, \not\leq)$

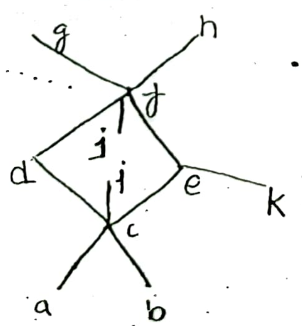
1. Maximal element: An element $a \in S$ is maximal iff No $b \in S$ exist such that $a \leq b$
[nothing above it]

2. Minimal element: An elmt $a \in S$ is minimal if No $b \in S$ exist such that $b \leq a$.
[no element below it]



Max $= \{d\}$

Min $= \{a\}$



Max $= \{i, g, h\}$

Min $= \{a, b, k, i\}$

- Both may or may not be unique but surely exist [for finite cases]

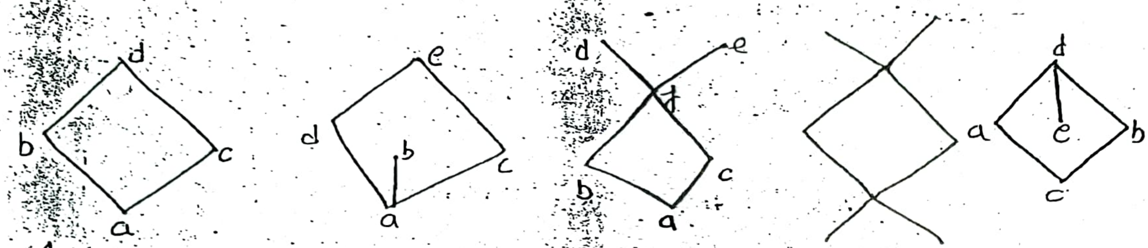
Greatest Element- element a is greatest iff $\forall b \in S$ $b \leq a$
denoted by 1) [Everything is below a]

- it is always unique
- It may or may not exist

Least element - a is least iff $\forall b \in S$, $a \leq b$
denoted by 0) (Everything is above a]

- unique, but may or may not exist.

Note- If a poset has both greatest & least elmt, it will be bounded



$1 = d$
$0 = a$

$1 \to$ does not exist
$0 = a$

$1 \to$ dont exist
$0 \to a$

$1, 0 \to$ dont exist

$1 \to d$
$0 \to$ not exist.

Upper Bound (A) - for $A \subseteq S$
$a \in UB(A)$ iff $\forall b \in A$, $b \leq a$
[above every element of A]

- not unique
- may or may not exist

Lower Bound (A) - for $A \subseteq S$
$a \in LB(A)$ iff $\forall b \in A$ $a \leq b$
[below every element of B]

- not unique
- may or may not exist

Lowest Upper Bound (A) for $a \in S$, $a = LUB(A)$ iff

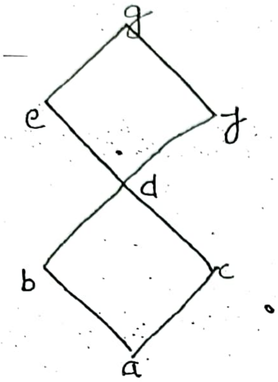
- denoted by $(a \vee b)$.

1, $a \in LB(A)$
2. $\forall b \in UB(A)$ $a \leq b$

- nearest neighbour
- unique
- may or may not exist

Greatest Lower bound (A) - $a \in S$ ~~such~~ than
$a = GLB(A)$ iff
1. $a \in LB(A)$
2. $\forall b \in LB(A)$ $b \leq a$

• denoted by $a \wedge b$.

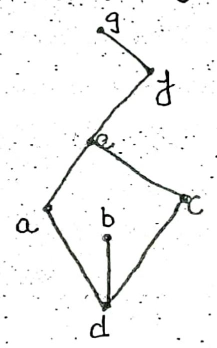


Let $A_1 = \{b, c\}$
$UB = \{d, e, f, g\}$
$LUB = \{d\}$
$LB = \{a\}$
$GLB = \{a\}$

Let $A_2 = \{b, d\}$
$UB = \{d, e, f, g\}$
$LUB = \{d\}$
$LB = \{b, a\}$
$GLB = \{b\}$

• the Lower bound or LUB or GLB for incomparable element never contains them.

Note- If two elements are comparable in a set that LUB & GLB will ~~be~~ the elements itself.

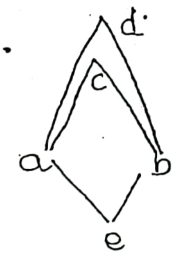


$A = \{a, e, f\}$
$LUB = \{g\}$
$GLB = \{a\}$

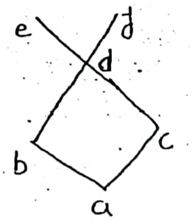
To find LUB, draw the arrow from all element of set the first point where they met is LUB (in upward direction)

To find GLB, draw the arrow in ~~up~~ down ward direction, the first point where arrow met is GLB.

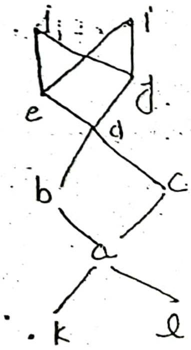
if the set does not have LUB ~~or GLB~~, the reason are,
(i) Not bounded
(ii) more than one exist



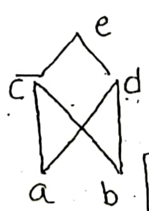
[No LUB exist]



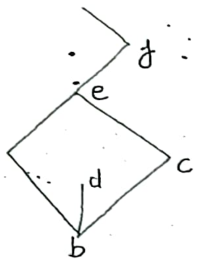
[Not upper bounded]
no (GLB)



No LUB [more than one bound]
No GLB



[No lower bound (LUB) exist]



$A = \{a, c, f\}$

~~LB{~~ ~~LUB{~~ $LUB(A) = \{f\}$

$GLB(A) = \{d\}$

Note- If LUB(S) exist for $(S, \leq)$, $LUB(S) = 1$ (Greatest element)

if GLB(S) exist for $(S, \leq)$, $GLB(S) = 0$ (Lowest element)

- $a+b = LUB(S)$
- $a \cdot b = GLB(S)$

] in boolean Algebra

Lattice -
- Lattice is one structure where every element has LUB & GLB.
- Greatest & least elmt must exist.
- It will satisfy closure, commutative, Associative property.
- It need not be bounded, complemented, distributive. [identity]

A Poset is a lattice iff $\forall a, b \in S$ $LUB(a, b)$ & $GLB(a, b)$ both exist & belongs to S.
$(S, \leq)$

i.e. $\forall a, b$ $a \vee b \in S$

$a \wedge b \in S$

Note- 1. $SUP[a, b] = LUB[a, b] = a \vee b = a + b = Join[a, b] = A \cup B$
Supremum

2. $Inf[a, b] = GLB[a, b] = a \wedge b = a \cdot b = Meet[a, b] = A \cap B$
Infimum

Standard Lattice-

Every Toset is a Lattice.

$(Z, \leq)$ $(R, \leq)$ → Always a lattice because for every pair of element LUB or GLB exist.

$(\{1, 3, 5, 10, 12\}, \leq)$ → Always a Lattice.

$(S, \subseteq)$ → Always a lattice

$(P(S), \subseteq)$ → Always a lattice

4. $(S, \subseteq) \rightarrow$ $LUB[A,B] = A \cup B$ $\quad \boxed{A, B \subseteq S}$

$GLB[A,B] = A \cap B$

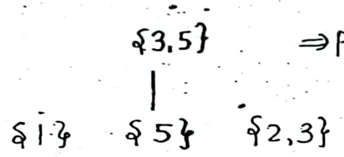
for $A = \{1,2\}$ $B = \{3,5\}$

$LUB[A,B] = \{1,2,3,5\}$

$GLB[A,B] = \phi$

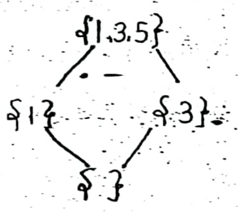
Same case happens with the $(P(S), \subseteq)$

Ques. $(\{\{1\}\{5\}\{2,3\}\{3,5\}\}, \subseteq)$



$\Rightarrow$ Poset but not lattice

Ques $(\{\{\}\{1\}\{3\}\{1,3,5\}\}, \subseteq)$



[$A \cup B = LUB(A,B)$
$A \cap B = GLB(A,B)$
True for $(S, \subseteq)$ & $(P(S), \subseteq)$
but not for a random set given]

5. $(Z^+, /) \Rightarrow$ [$LUB(a,b) = LCM(a,b)$, $GLB(a,b) = gcd(a,b)$] always exist

Hence, it will be lattice.

What is $2 \vee 3 \wedge 5$?

$\Rightarrow 2 \vee (3 \wedge 5)$ [due to precedence]

$\Rightarrow 2 \vee 1$

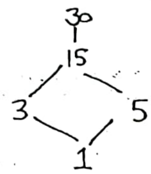
$\Rightarrow 2$

6. $(D_n, /) \Rightarrow$ Always a lattice

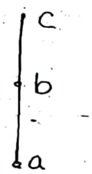
- $LUB(a,b) = LCM(a,b)$
$GLB(a,b) = gcd(a,b)$] Always exist

[if a,b are divisor of n than LCM(a,b) will also be divisor of n & always exist
Also, gcd(a,b) will exist]

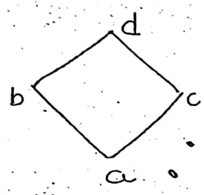
Ques $(\{1, 3, 5, 15, 30\}, /)$



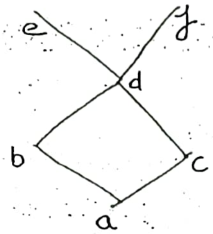
LUB GLB exist for each element
Hence it is Lattice.



Lattice



Lattice



Not a Lattice
$LUB(e, f)$ = not exist

For random set,
to check whether it is lattice or not
check for all non-comparable pairs:

for a Random set $(S, /)$
LUB may ~~not~~ or may not be $LCM(a, b)$
GLB ———————— $HCF(a, b)$

$(R \times R$, product partial order$)$

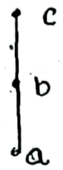
$$\boxed{\begin{aligned} LUB[(x_1, y_1)(x_2, y_2)] &= [\max(x_1, x_2), \max(y_1, y_2)] \\ GLB[(x_1, y_1), (x_2, y_2)] &= [\min(x_1, x_2), \min(y_1, y_2)] \end{aligned}}$$

Hence, it will be a Lattice.

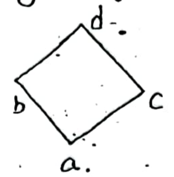
$(Z \times Z$, product partial order$)$

It will also be a lattice (same as $R \times R$)

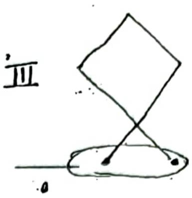
Which of the following represent Lattice ?

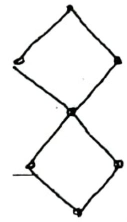
I 

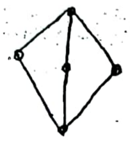
Lattice

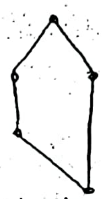
II 

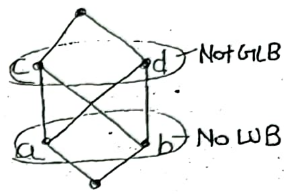
Lattice

III 
Not a lattice

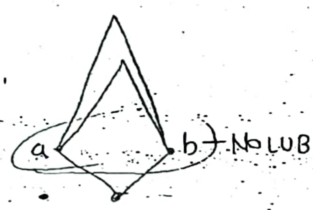
IV 
Lattice

V 
Lattice

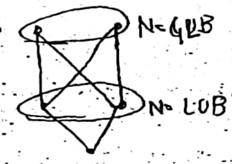
VI 
Lattice [Pentagonal Lattice]

VII. 

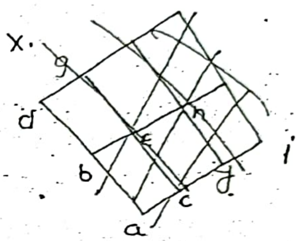
Not a Lattice

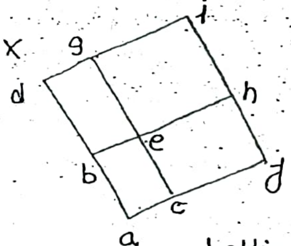
VIII. 

Not a lattice

IX. 

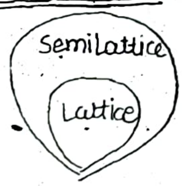
Not a lattice

X. 

X 

Lattice

Semi Lattice:



A poset is semilattice iff all LUB[a,b] exist OR all GLB exist.

[$\forall a,b \in S$ LUB(a,b) exist] OR [$\forall a,b \in S$ GLB(a,b) exist]

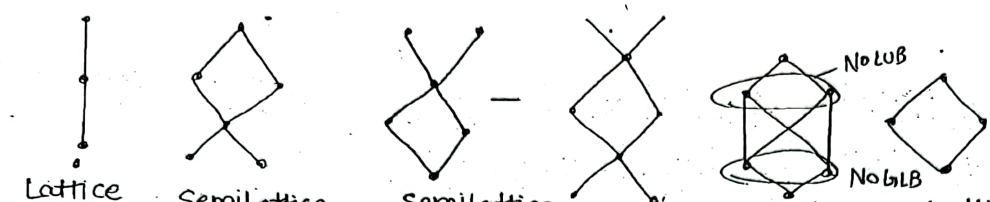
- all pair should have LUB or GLB.

- A Lattice is always a semiLattice.

Ques- Which is true for
$\forall a,b \in S\,(\text{LUB}(a,b) \vee \text{GLB}(a,b))$

(a) Lattice
(b) Semilattice
(c) None

Which of the following is semilattice?



Lattice — Join SL as well as MSL

Semilattice [Join Semi Lattice]

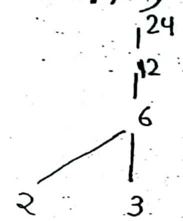
Semilattice [Meet SL]

Not a Semilattice

Not a Semilattice

Lattice — JSL as well MSL.

Ques ({2, 3, 6, 12, 24}, /)



$\Rightarrow$ Join SemiLattice.

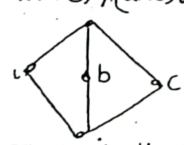
Types of Semilattice-

1. JSL (Join Semilattice) – if all LUB exist
2. MSL (Meet Semilattice) $\rightarrow$ if all GLB exist

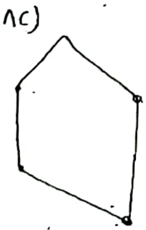
Properties of Lattice-

- Closure- $\forall a, b \in S$ $\quad a \vee b \in S$, $\quad a \wedge b \in S$
- Commutative $\forall a, b \in S$ $\begin{bmatrix} a \vee b = b \vee a \\ a \wedge b = b \wedge a \end{bmatrix}$
- Associative $\forall a, b, c \in S$ $\begin{bmatrix} a \vee (b \vee c) = (a \vee b) \vee c \\ a \wedge (b \wedge c) = (a \wedge b) \wedge c \end{bmatrix}$

$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$



[Kite Lattice]



[Lattice but not distributive]

A Lattice may or may not be distributive, bounded & complemented

- $(Z, \leq) \rightarrow$ Lattice but not bounded (1 or 0)
- $(D_{20}, /) \rightarrow$ ~~Not com~~ Lattice but not complemented.

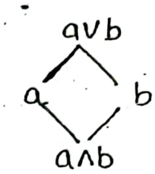
Idempotent Law - lattice satisfy this law

$\forall a \in S \quad a \vee a = a$

$a \wedge a = a$

Law of absorption - $a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$



[Lattice may or may not satisfy Demorgan law, Double complement & Domination Law.]

Properties:

1. **Consistency Law** - for $\forall a, b \in S$ $\quad a \leq a \vee b \quad a \wedge b \leq a$

$b \leq a \vee b \quad a \wedge b \leq b$

2) a & b are comparable iff $\begin{bmatrix} a \vee b = b \\ \& \; a \wedge b = a \end{bmatrix}$

$(a \leq b)$

Ques - if $x \leq y, y \leq z$

What is $x \wedge z$?

$x \vee z$?

$x \leq y, y \leq z \Rightarrow x \leq z$

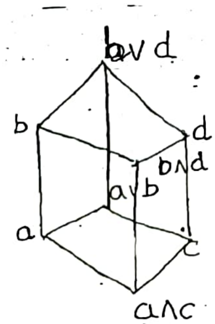
$\therefore \begin{bmatrix} x \wedge z = x \\ x \vee z = z \end{bmatrix}$

Ques - Which is true?

1. $a \vee b = b \Rightarrow a \leq b$ [True] [only 1 is sufficient to prove it true. If $a \vee b = b$ then surely a, b are related]
2. $a \vee b = b \Rightarrow b \leq a$ [False]

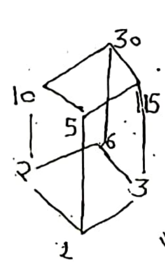
3) if $a \leq b$ & $c \leq d$ then $a \vee b \leq c \vee d$

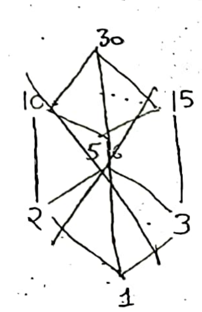
& $a \wedge b \leq c \wedge d$



$\Rightarrow$ always a Lattice

Ex $(D_{30}, /)$

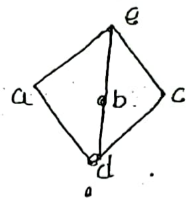


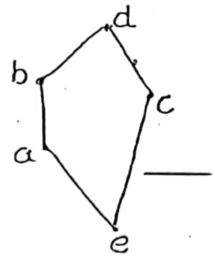


1. Lattice always satisfy distributive Inequality.

1. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

2. $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$





e greater or equal a, b. 1
d greater or equal a, b 1
a, b. ~~greater~~ smaller or equal e.d
greater or equal b, c.

Types of Lattice

Bounded Lattice A Lattice $(L, \leq)$ is bounded iff $0, 1 \in L$.
[Least & greatest elmt exist]

- A Finite Lattice is always Bounded.
  Let $S = \{a_1, a_2, a_3, a_4, \ldots, a_n\}$

  $a_1 \vee a_2 \vee a_3 \ldots a_n = 1$

  $a_1 \wedge a_2 \wedge a_3 \ldots a_n = 0$

~~Property~~

Some infinite Lattice are also bounded.

S = Real No b/w 0, 1 $(S, \leq)$

$0 \leq x \leq 1$

it is infinite but bounded.

- An uncountable infinite Lattice can be bounded.
- A countable infinite Lattice can never be bounded.

A Lattice bounded iff $\forall a$ $\boxed{0 \leq a \leq 1}$

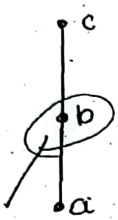
Complement Lattice: Lattice $(L, \leq)$ is complemented iff

$\forall a \; \exists a'$ $a + a' = 1$

$a \cdot a' = 0$

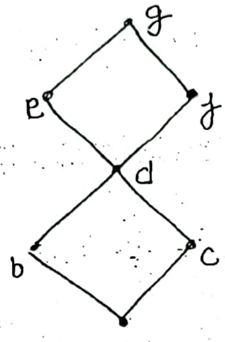
Property-
- $0' = 1$
- $1' = 0$
- A complemented Lattice is always bounded. $CL \Rightarrow BL$

- if a is not least elmt (0) or greatest elmt (1), than, a' can not be comparable to a.
- 0 & 1 are comparable to each other & complement of each other.



Not element to become b'

$\nexists$ Not a CL



Not a CL

complement of b is only cand. is c

$b \vee c = d$

$\therefore$ b' does not exist

c' ———

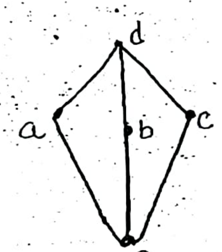
e' – not exist

f' – not exist

$\therefore$ Hence it is not a complemented lattice.

Theorem- A Toset can never be a complemented lattice if $n \geq 3$

A Toset is a complemented lattice $\Rightarrow$ Toset has 2 element

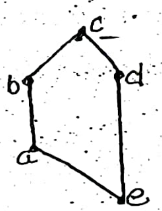


complemented lattice

$a' = b, c$

$b' = a, c$

$c' = a, b$ [can have more than one complement]

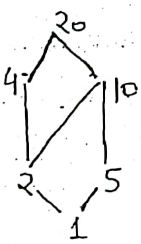


$a' = d$

$b' = d$

$d' = a, b$

$(D_{20}, /) \Rightarrow$



$2' =$ does not exist

Distributive Lattice A Lattice $(L, \leq)$ is DL

iff

$\forall a,b,c \in S \quad a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$

$\qquad\qquad\quad a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$ — in BA.

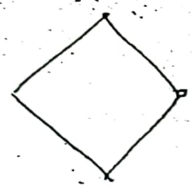
ich is true for a distributive lattice? _

1) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
) $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$
) $(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$ [All are true]
$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

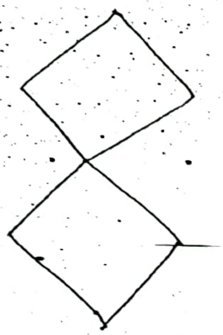
eck whether it is distributive lattice or Not?

eorem1. A Lattice is distributive iff it does not contain kite Lattice or Pentagonal lattice as sublattice-

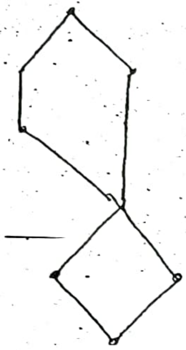
— A Lattice is not distribute iff it contain kite lattice or pentagonal Lattice.



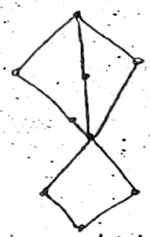
Distributive Lattice as well as boolean Algebra



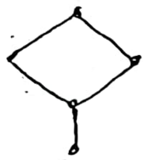
Distributive



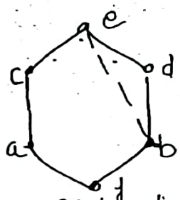
Not a distributive lattice



Not a distributive Lattice



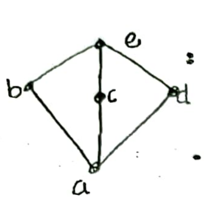
Distributive Lattice



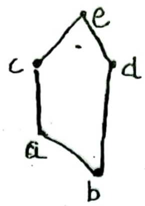
Not a distributive Lattice

$d' = a, c$
$a' = d, b$
$c' = d, b$
$b' = a, c$
} Multiple complements. Hence not distributive.

Theorem 2- A complemented Lattice is distributive ⇒ Complement for every element is unique.

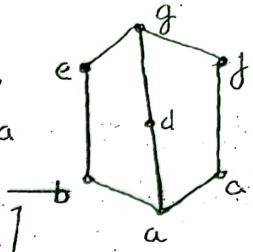


$a' = c, d$
$c' = b, d$
$d' = b, c$
$\Rightarrow$ Not distributive



$a' = d$,
$c' = d$
$d' = c, a$

[Not a distributive Lattice]



$b' = d, f, c$
Not a distributive Lattice

Note- · A complemented distributive lattice is always Boolean Algebra
· Boolean Algebra, every elmt has unique complement.
· If a complemented lattice have unique complement, it may or may not be distributive.

Boolean Algebra

A Lattice is BA iff (i) distributive.
(ii) Bounded.
(iii) complemented.

· Hasse diagram is denoted by Q_n.

1. $(D_n, /)$ is a boolean Algebra if n can be divided into distinct prime & It is isomorphic to $(P(S), \subseteq)$

Let x be the No of distinct primes

$$\left[\begin{array}{l} \text{No of edges} = x \cdot 2^{x-1} \\ \text{No of vertices} = 2^x \end{array}\right]$$

2. $a \le b$ iff $ab' = 0$

Ques In $(D_{30}, /)$ what is $(5 + 6 \cdot 3')$?

Note- In $(D_n, /)$, $\forall a$, $\boxed{a' = n/a}$

$5 + 6 \cdot 3' = 5 + 6 \cdot 3'$
$= 5 + 6 \cdot 10$
$= 5 + 2$
$= 10$

$3' = 30/3$

$a \cdot b = \gcd(a, b)$
$a + b = \mathrm{LCM}(a, b)$

Proof - let us compare it with Set theory,
$A \subseteq B$ if $A \cap \bar{B} = \phi$
True

Ques: A ... when ~~...~~ ... of BA?

1. $(c+a)(c+b') = c+ab' = c$

2. $c \vee (a \wedge b') = c \vee 0 = c$

Property 3: ~~Both~~ Left cancellation & right cancellation does not hold in BA.

- $a+b = a+c \not\Rightarrow b=c$

[it does not hold]

- $b+a = c+a \not\Rightarrow b=c$
- $a \cdot b = a \cdot c \not\Rightarrow b=c$
- $b \cdot a = c \cdot a \not\Rightarrow b=c$

> Note- If $a+b=a+c$ and $a \cdot b = ac$ $\Rightarrow b=c$

16-Sept

Graphs

Graph theory

1 Definition of graph, Types of graph
2 Imp{ Degree of a vertex
3 Special graph
4 Graph Representation
5 Imp { Graph operations: $\cup, \cap, \overline{G}, G_1 - G_2, G_1 \oplus G_2$ Imp
6 Isomorphism of Graph
7 Connectivity of Graph
1. Connected & Disconnected Graph Imp
2. Cut Edge, Cut vertex, cutset Imp
3. Vertex connectivity, Edge connectivity [for undirected graph]
4. ~~Stor~~ Strongly connected, Unilaterally connected, Weakly connected & Disconnected. [for directed graph]

applications: → Advance Graph theory

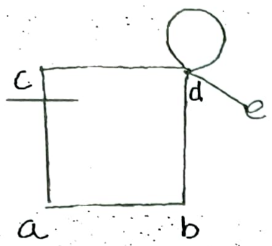
1 • Euler's Graph, hamiltonian Graph
2 M.Imp • Planar Graph (Most Imp)
3 • Trees
4 imp • Enumeration of Graph (Counting)
5 Graph No (i) chromatic No (Imp).
(ii) Independence No
(iii) Domination No
(iv) Matching No
(v) Covering No

$$\boxed{\text{Graph Theory} = 7 + 5}$$

Application → Railway logistics, transportation/optimization, O.R (operation Research), W.L (war logistics)

- **Graph-** (used to solve constraint optimization problem)

A Graph $G(V, E)$ is set of vertices & edges



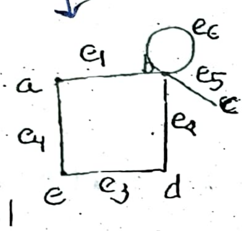
$V = \{a, b, c, d, e\}$

$E = \{\{a,b\}\ \{a,c\}\ \{c,d\}\ \{b,d\}, \{d,e\}\ \{d,d\}\}$

↳ for directed graph, it will become a ordered pair.

Undirected $= \{\{a,b\}\}$
Directed $= \{(a,b)\}$

- **Edge Labelled graph-**



$G(V, E, f)$

$f(e_1) = \{a, b\}$

$f(e_2) = \{b, d\}$

$f(e_3) = \{d, e\}$

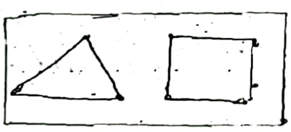
- Two vertices are adjacent if there is an edge b/w them
  - (e,d) are adjacent $\quad e \xrightarrow{e} d$
- Every edge is incident on exactly two vertices, necessarily not distinct.
- **Order of Graph-** the No of vertices in a graph represents order of graph

$\boxed{O(G) = 5}$

- **Size of Graph-** No of edges in a graph

$\boxed{\cancel{e(G) = 6}}$ $\quad Size(G) = e = 6$

- **Connected components (K)** No of connected subgraph in a given graph



$O(G) = 7$
$Size(G) = 7$
$K = 2$

No of components in connected graph = 1

No of components in NULL graph with n vertices = n

- for a graph, edge set can be empty but vertex set cannot be empty.

Book :- Canathramem

Types of Graph

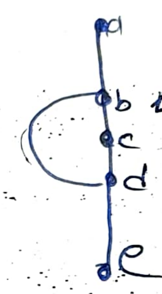
- Graph
  - Undirected Graph
    - Simple Graph (S.G)
    - Multi Graph (M.G)
    - Pseudo-Graph (P.G)
  - Directed Graph.
    - Simple Directed Graph
    - Multi Graph
    - Pseudo graph

Simple Graph-
- No self loops
- No Multiple edge

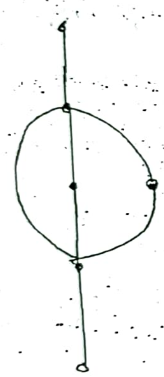
every S.G is M.G and every M.G is a P.G ..

Multi Graph-
- Multiple edges are allowed.
- Self loops are not allowed.

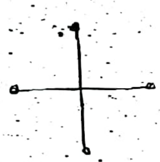
Pseudo Graph-
- Self loops are allowed.
- Multiple edges are allowed.



Simple graph



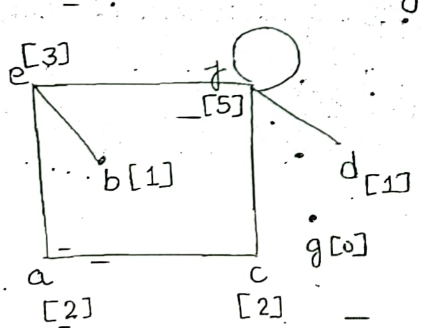
⇒ Simple graph



Disconnected graph

Degree of a Vertex- For vertex v,

Deg (v) = No of edges incident upon v
(self loop will be counted twice).



Lone vertex : A vertex with degree 0

Pendant Vertex : A vertex with degree 1 (leaf nodes)

Leaf is a pendent vertex in tree.

Degree Sequence- • writting the degree of the graph in some sequence [either increasing order or decreasing order]

$= [0, 1, 1, 2, 2, 3, 5]$

or

$[5, 3, 2, 2, 1, 1, 0]$

• it is always unique for a graph.

• Every degree sequence represents a unique graph: False.

• Degree sequence for a simple graph is known as graphical sequence.

MiniMum Degree of Graph (δ) - $\delta = 0$ [unique]

MaxiMum Degree of graph (Δ) - $\Delta = n-1$ (for complete graph)

Note- if $\delta = \Delta$ then graph is known as Regular Graph.

$\boxed{\delta = \Delta = K \Rightarrow K\text{-Regular Graph}}$

→ 2 Regular Graph

only thee algo.

1 • Hand Shaking Theorem

2 • δ, Δ,

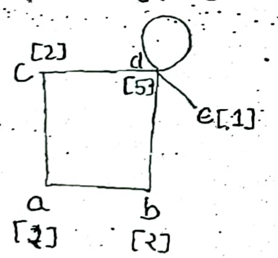
3 • Havel Hacime

Theorems -

1. Handshaking Theorem:

$$\boxed{\sum_{\forall V \in G} \deg(V) = 2e}$$

the sum of degree of all vertices is always twice of No of edges.



∴ Sum of degree = 12

∴ No of edges = 6

Que: 1, 1, 2, 2, 3, 3, 3, 4, 5

$n = 9$

$e = ?$

$e = \left[\frac{\deg}{2}\right] = \frac{24}{2} = \underline{12}$ Ans

Note- Sum of degree of a graph is always even.

corollary: 1. No graph can have odd No of vertices of odd degree

Pf: Sum of odd degree vertices + Sum of even degree vertices $= 2e$

Sum of odd degree vertices $= 2e - $ Even

$=$ Even

[No of odd degree = Even]

$\boxed{\text{In every graph, the no of vertices of odd degree is always even.}}$

ues- Which is true?

a) the No of even degree vertices is always even.

b) the No of even degree vertices is always odd.

c) the No of odd degree vertices is always even.

d) No No of odd degree vertices is always odd.

es- Is Graph possible with this degree sequence (1, 1, 2, 2, 3, 4, 5, 7)

↳ No [No of vertices of odd degree = odd]

- $\boxed{\text{Max No of edges in simple graph is } \frac{n(n-1)}{2}}$

for n vertices, to get max No of edges every vertex must be connected to every other vertex in the graph

∴ Degree of each vertices = $n-1$

$$\sum_{v=1}^{n}(n-1) = 2e$$

$$n(n-1) = 2e$$

$$\Rightarrow e = \frac{n(n-1)}{2} \quad [{}^nC_2]$$

- No of edges in complete graph with n vertices

$$\left[e(K_n) = \frac{n(n-1)}{2}\right]$$

K_n- complete graph with n vertices.

- A K regular graph with n vertices has $\frac{nK}{2}$ edges

K=3
n=6
$=\frac{\cancel{6}\times 3}{2}=9$

- $$\left[e(K\text{-regular graph}) = \frac{nK}{2}\right]$$

for n-1 regular graph

$$\left[e(n\text{-}1 \text{ reg.}) = \frac{n(n-1)}{2}\right]$$ ⇒ complete graph.

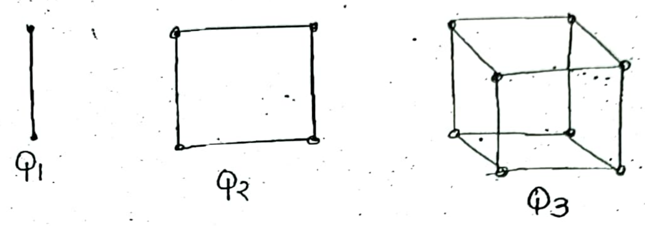
Note- A complete graph is always regular but not vice-versa.

6) No Graph of odd order can be K-regular when k is odd.

If it becomes true, No of odd order vertices will be odd in the graph

• Graph of odd order can be K-regular when k is even
↳ True.

7)



8) for Q_n Graph, No of vertices $= 2^n$

No of edges $= n \cdot 2^{n-1}$

As we know that, Q_n is always n regular.

$$\boxed{\text{No of edges} = \frac{n * \text{No of vertices}}{2} = n \times 2^{n-1}}$$

Ques- In a tree, every vertex has got degree 1, 2, 3, ... k.

No of vertices of degree $2 = n_2$

⋮

No of vertices of degree $k = n_k$

∴ How many leaf nodes are there (with degree 1) Let x

$$\Sigma \text{ degree} = 2e$$

$$\text{degree } 1 + \text{degree } 2 + \text{degree } 3 + \ldots = 2e$$

$$1 * x + 2 * n_2 + 3 * n_3 + \ldots + k n_k = 2 * e$$

$$x = 2e - [2n_2 + 3n_3 + \ldots + k n_k]$$

In a tree $[e = n - 1]$

$$e = n_2 + n_3 + n_4 + \ldots n_k + x - 1$$

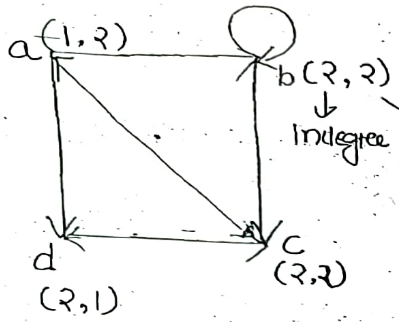
$$x = 2n_2 + 2n_3 + 2n_4 + 2n_k + 2x - 2 - 2n_2 - 3n_3 - \ldots k n_k$$

$$= 2x - n_3 - 2n_4 - 3n_5 \ldots (k-2)n_k - 2$$

$$\boxed{x = n_3 + 2n_4 + 3n_5 + \ldots + (k-2)n_k + 2}$$

Handshaking Theorem for Digraph:

$$\boxed{\sum_{\forall v \in G} \text{indeg}(v) = \sum_{\forall v \in G} \text{outdegree}(v) = e}$$



No of indegree = No of out degree = No of edges

self loop counts for 1 indegree and 1 out degree.

i the sequence of degree is given as

$(1,2)\ (2,2)\ (2,y)\ (x,2)$

What is the rel^n b/w x & y

$1+2+2+x = 2+2+2+y$

$\boxed{x = y+1}$

HAVELL-HAKIMI THEOREM-

It is used to check whether a ~~graphical seque~~ degree sequence corresponds to graphical sequence.

[graphical sequence of a simple graph]

~~A decreas~~

~~A degree sequence $|d_1\ d_2\ d_3 \ldots d_n|$ in non-increasing order is Graphical (decreasing order) Sequence iff $|d_1-1, d_2-1, d_3-1, \ldots d_n-1|$ is a graphical sequence.~~

A degree sequence $|d_1\ d_2\ d_3 \ldots d_n|$ in decreasing order is a graphical sequence iff $|d_2-1, d_3-1, \ldots d_{d_1+1}-1, d_{d_1+2}, d_{d_1+3} \ldots d_n|$ is a graphical sequence.

$(1, 1, 1, 1, 0, 0, 0, 0)$

↳ Simple Graph ∴ A Graphical sequence.

- Repeatedly check until the sequence of 1's & 0's is obtained.

Note-
It is possible that if no of odd vertices is even & no graph exists for it.

Graph exist $\Rightarrow$ No of odd vertices is even
[one way]

Ques- Which of the following is graphical sequence?

I (5, 5, 5, 4, 4, 3, 3, 2, 2, 1)

II (4, 4, 3, 3, 3, 2, 1)

III (4, 4, 3, 3, 3, 2, 2)
↳ Not possible to be a graph

A sequence with even no of odd degree vertices is not GS iff
a) No of vertices is less than degree of vertex cutted
Ex (5, 4, 3, 3, 3)
⇓ Apply Havel Hakimi
(3, 2, 2, 2 __)
↓ No one to subtract
Hence, not a Graphical Sequence

b) force to put -1.
Ex (~~6~~, 4, 3, 3, 3, 0, 0, 0, 0)
⇓ Apply Havell Hakimi
(3, 2, 2, 2, -1, 0, 0, 0)
Hence not a graphical sequence.

- it is possible to have a GS with 1's only.

Ques- (5, 5, 4, 4, 3, 2, 2, 1)
Check whether it is GS or not?
(i) (check it should have even no of odd vertices (degree))
(ii) it should be in order (descending)
(iii) remove 1'st vertex & delete its degree
(4, 3, 3, 2, 1, 2, 1)

$\equiv (\cancel{4}, 3, 3, 2, 2, 1, 1)$

$\equiv (\cancel{2}, 2, 1, 1, 1, 1)$

$\equiv (1, 0, 1, 1, 1)$

$\equiv (1, 1, 1, 1, 0)$

Hence it is a graphical sequence.

Result

Corollary- Any graphical sequence must have atleast one repitition.

Imp

Ex- $(7, 6, 5, 4, 3, 2, 1)$

to remove 7

(7 No should be there to subtract 7)

but if no repitition is there

only 6 is there to subtract 7

$\therefore$ for a Sequence to be GS, it should have atleast one repitition.

δ, Δ theorem

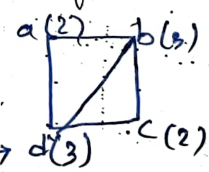
$$\boxed{\delta \leq \frac{2e}{n} \leq \Delta} \rightarrow \text{min} < \text{avg} < \text{max}$$

$\Downarrow$

Average degree Need not to be integer

$$\text{Avg. Degree (per vertex)} = \frac{\text{total degree}}{\text{No of vertices}} = \frac{2e}{n}$$

Ex:-



$2 \leq 2.5 \leq 3$

Ques- if order of graph $= 10$ & $\delta = 3$

What is the min No of edges?

$$\delta \leq \frac{2e}{n} \Rightarrow e \geq \frac{n\delta}{2} = 15$$

$$\boxed{e \geq 15}$$

- if $\Delta = 3$

$$\frac{2e}{n} \leq \Delta$$

$$e \leq \frac{\Delta n}{3} \Rightarrow e \leq 15$$

Ques- What is the max. value of δ with order 10 & size 16?

$$\delta \leq \frac{2e}{n} \qquad \delta \leq \frac{2 * 16}{10} \qquad \delta \leq 3.2$$

$\therefore$ δ is atmost $= 3$

Note-

Max value of $\delta = \left\lfloor \frac{2e}{n} \right\rfloor$

Min. Value of $\Delta = \left\lceil \frac{2e}{n} \right\rceil$

- $VC \leq EC \leq \delta \leq \frac{2e}{n} \leq \Delta$

Special Graph: (Imp)

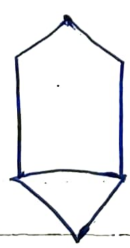
$\left(\frac{n(n-1)}{2}\right)$

Chromatic No — Diameter

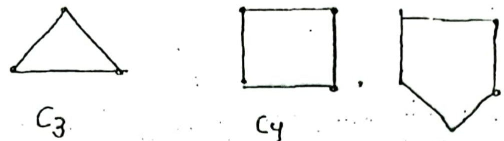
	Name	Notation	Vertices	Edges	Definitition	K_G	Diameter
1.	NULL GRAPH	ϕ_n	n	0	A Graph is NULL if $\lvert e \rvert = 0$	1	∞
2.	COMPLETE GRAPH	K_n	n	$^nC_2$	G is K_n iff $\lvert E \rvert = {}^nC_2$	n	1
3.	REGULAR GRAPH	K-Reg	n	$\frac{n \cdot K}{2}$	G is K regular iff $\delta = \Delta = K$		
4.	CYCLE GRAPH	C_n	n	n	G is cycle iff it is a polygen	$\begin{cases} 2 & n \in \text{even} \\ 3 & n \in \text{odd} \end{cases}$	$\left\lfloor \frac{n}{2} \right\rfloor$
5.	WHEEL GRAPH	W_n	n	$2(n-1)$	cycle graph with one vertex connected to every other.	$\begin{cases} 3 & n = \text{odd} \\ 4 & n = \text{even} \end{cases}$	$\begin{cases} 1, & n = 4 \\ 2, & n \geq 5 \end{cases}$
6.	n-CUBES	Q_n	2^n	$n \cdot 2^{n-1}$	G is n-cube if it is a boolean Algebra	$Q_n = 2$	n
7.	BIPARTITE GRAPH	-	-	-	A G is bipartite if vertex set can be partition into sets such that no edge inside the vertex	for non-null $K(B_p) = 2$	≥ 2
8.	COMPLETE BIPARTITE GRAPH	$K_{m,n} = K_{n,m}$	$m+n$	mn	$e \in G \Leftrightarrow V_1 - V_2$	$K = 2$.	$2, m \geq 3$

If you find any tringle, Pentago (odd) sequence then this will surely not be a biparted graph.

X ;



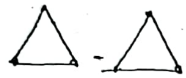
Cycle graph-



C_3 C_4 C_5

- C_n exist $n \geq 3$
- $C_n = 2$ is not allowed in simple graph.

$\boxed{K_3 = C_3}$



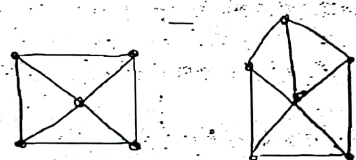
Note-1. A complete graph is cycle is $n=3$

2) A cyclic graph is different from cycle graph.
(it contains a cycle) (Graph itself is one single cycle)

3) A cycle graph is always cyclic.

cycle graph $\Longrightarrow$ cyclic graph
[one way theorem]

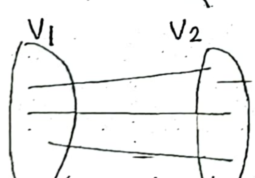
Wheel Graph-



W_5 W_6

exist for $n \geq 4$.

Bipartite Graph- A Graph G if its vertex set can be partitioned into two set V_1 & V_2 such that No edges inside V_1 & inside V_2 edges are from V_1 to V_2 ~~or~~ ~~vice versa~~



$[e \in G$ iff $V_1 \rightarrow V_2]$ — (it may or may not have that edge)

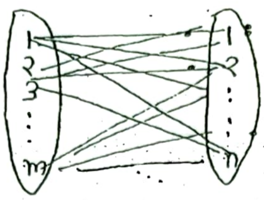
V_1 & V_2 must be disjoint

Complete Bipartite Graph • can be partition into vertex set V_1 & V_2 disjoint

&&

$[e \in G$ (iff) $V_1 — V_2]$
$\forall v \in V_1$

No of edges-

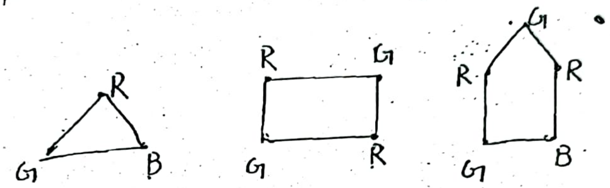


$\Rightarrow$ No of edges $= mn$

Chromatic No - No of color required for proper coloring of a graph [No two adjacent vertex should have same color]

Note- A graph has chromatic No 1 iff it is a null ~~graph~~ graph

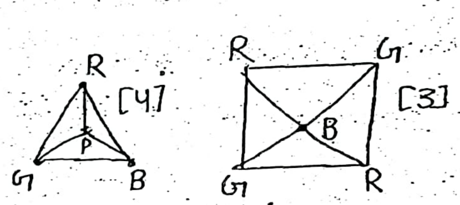
Cycle graph-



[3] [2] [3]

$\therefore$ chromatic No $k(G) = 2 + n \bmod 2$

Wheel Graph-



[4] [3]

Chromatic No $k(G) = 3 + (n-1) \bmod 2$
$\equiv 3 + (n+1) \bmod 2$
$\equiv 4 - n \bmod 2$

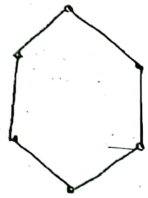
Bipartite Graph- since vertex set can be partitioned into two set V_1 & V_2 & no edges inside it $\therefore$ then it can be colored with 2 color.

~~Set~~ Special case- if ~~no~~ ϕ_n $n \geq 2$
$\therefore$ chromatic No $= 2$
Since Null graph is also a bipartite graph.

$$\text{chromatic No} = \begin{cases} 1 & \text{if null graph} \\ 2 & \text{if Non-Null graph} \end{cases}$$

- Every Null graph is bipartite. False (ϕ_1)

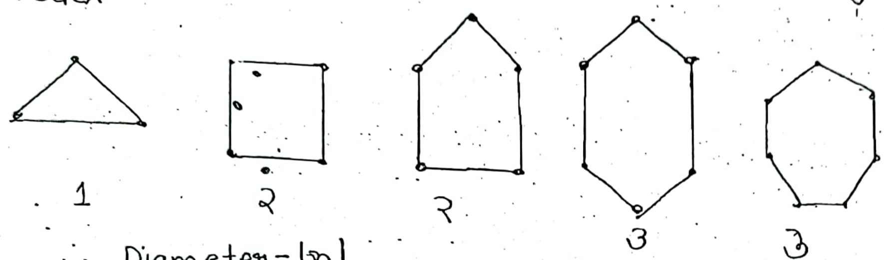
Iameter- it is length of shortest path between two farthest points.



Diameter = 3

~~full~~ for Null Graph, diameter is ∞

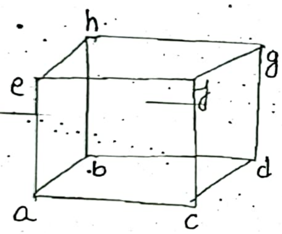
for complete graph diameter = 1 bcoz there exist an edge b/w every vertex.

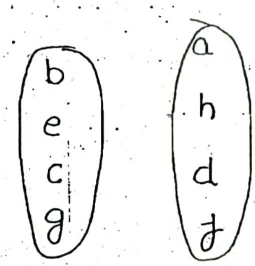


$\therefore$ Diameter $= \left\lfloor \frac{n}{2} \right\rfloor$.

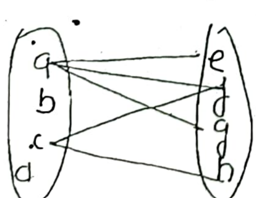
hromatic No of C_n - it is always 2.

Theorem- A Graph G is bipartite if and only if G has no odd cycles.





f a graph is bipartite no odd cycle allowed

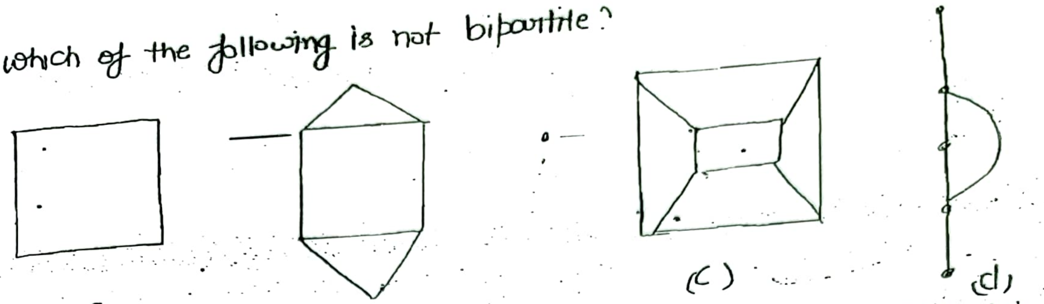


No odd cycles allowed.

Note- A graph G is bipartite if and only if it has even cycle.
[False]
it may or may not have even cycle.

Theorem 2- A non-null graph is bipartite if and only if it is bichromatic.

Ques- which of the following is not bipartite?

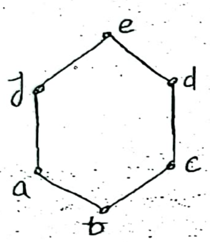


(a)

(b) it is not bipartite graph

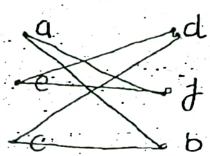
(c)

(d) Not a bipartite graph.



$V_1 = \{a, e, c\}$

$V_2 = \{f, b, d\}$



Note- A Acyclic graph is always bipartite.
A tree with two or more vertices is always bipartite & hence bichromatic.

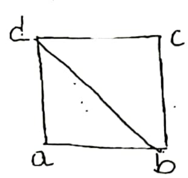
- Diameter of a complete bipartite graph is 2.
  $\therefore$ Ratio(Diameter/ chromatic No.) = 1 : 1

- Note- We cannot predict the diameter of bipartite graph it will be atleast 2 & atmost ∞ (No path)

Graph Representation :

1. Adjacency Matrix (for a simple graph)
2. Adjacency list. • cant represent a multigraph
3. Incidence Matrix. [when edges are labelled]

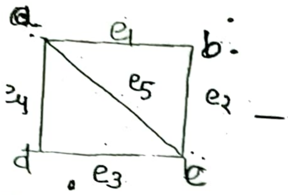
1. Adjacency Matrix : Size of Matrix = $|V| \times |V|$



$$\begin{array}{c|cccc} & a & b & c & d \\ a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{array}$$

Incidence Matrix.

Size = $|V| \times |E|$

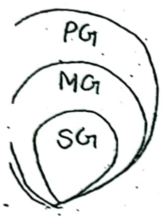


$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Adj. Matrix

Properties-

1. Any edge in the ~~adja~~ adj graph other than self loop corresponds to two 1's in matrix.
2. if there are any multiple edges present in graph, it will not be boolean matrix.
3. [If diagonals are zero & a boolean matrix ⇒ G is a simple graph.]
4. for an undirected graph, Adj Matrix is symmetric.
5. Adjacency Matrix can represent SG, MG, PG.



Note- $A_{n \times n} = \{a_{ij}\}$

$$\sum_{\forall i \in n} \sum_{\forall j \in n} = 2 \times |E|$$

⇒ It is true only for Multigraph but not pseudograph. (bcoz self loop will correspond to 1 entry in the table)

6. Row Sum & Column Sum represent the degree of vertex in adjacency matrix.

for pseudo graph.

$$A_{n \times n} = \begin{cases} a_{ij} & i \neq j \\ 2 \times a_{ij} & i \neq j \end{cases}$$

7. for a directed graph, Adjacency matrix may or may not be symmetric.
8. Row Sum in Undirected graph = Degree of vertex
 ———— Directed graph = out degree of vertex
 Column sum in Directed graph = Indegree

In Directed graph,

$$\sum_{\forall i \in n} \sum_{\forall j \in n} A = |E|$$

no of directed edge

true for every pseudo graph. (bcoz directed edge will count loop once)

Properties of Incidence Matrix :

1> each column will contain exactly 2 1's (not for self loops)

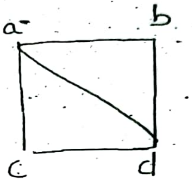
2> it is possible only for edge labelled undirected graph.

3> In self loop, column will contain a single 1.

4> Row sum corresponds to degree of a vertex.

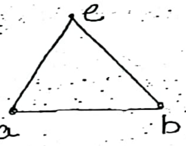
Adjacency List : · it is not possible for multigraph.

	Adj vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c.

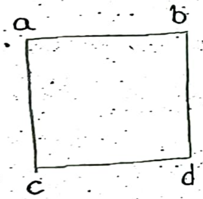


Graph operations-

Let G_1 =



G_2:

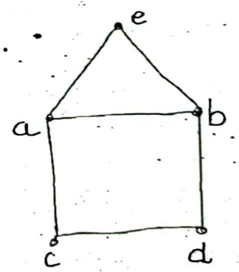


1. $G_1 \cup G_2$

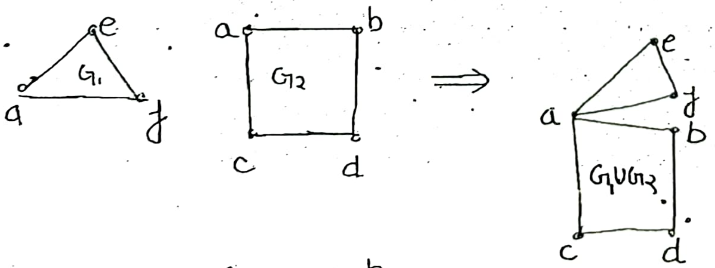
$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$

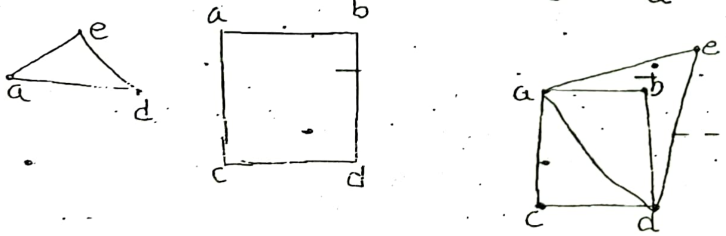
$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

· $V(G_1 \cup G_2) = \{a, b, c, d, e\}$



· it is always unique.





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1) each column will contain exactly 2 1's (not for self loops)
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3) In self loop, column will contain a single 1.
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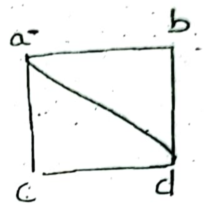
Adjacency List:

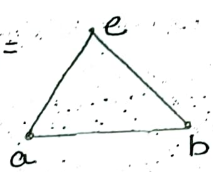
• it is not possible for multigraph.

	Adj vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c.

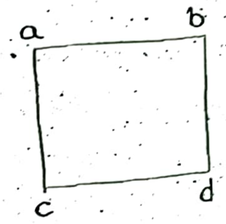


Graph operations-

Let G_1 =



G_2:

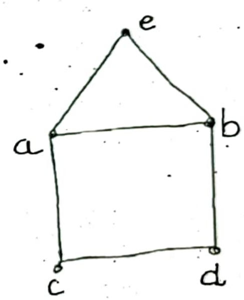


1. $G_1 \cup G_2$

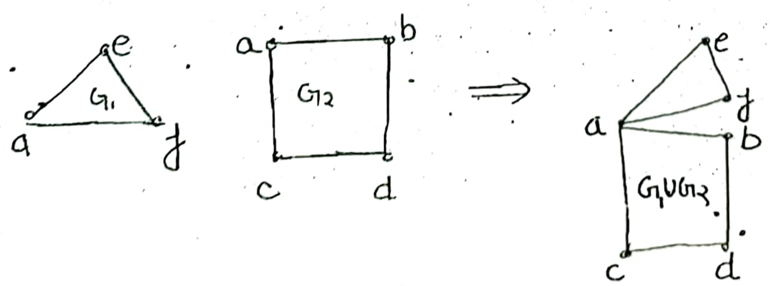
$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$

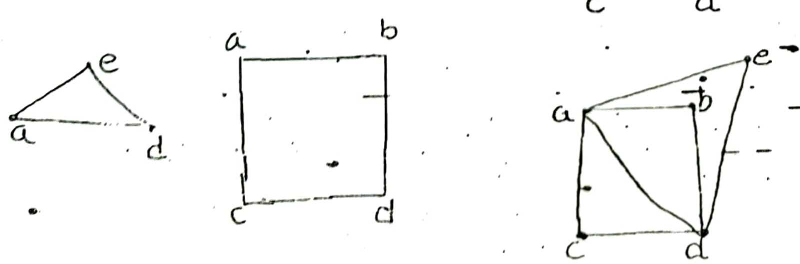
$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

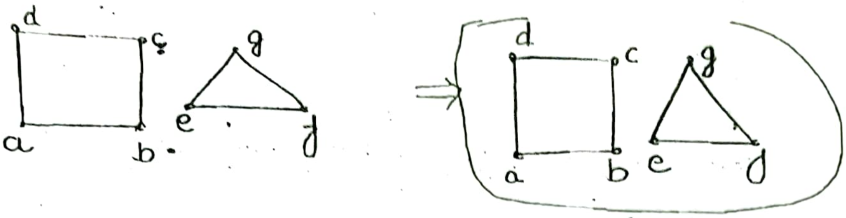
• $V(G_1 \cup G_2) = \{a, b, c, d, e\}$



• it is always unique.





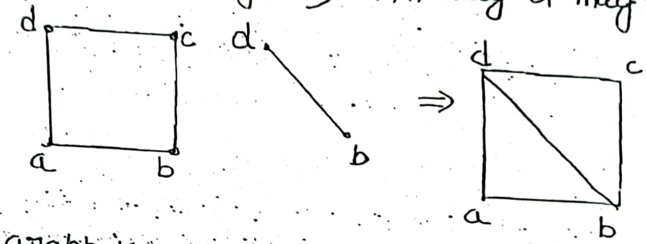


$G_1 \cup G_2$

Note-1. for two connected graph G_1 & G_2, $G_1 \cup G_2$ will be disconnected only if G_1 & G_2 are vertex-disjoint graph.

$V_1 \cap V_2 = \phi$

2. If the graph are edge disjoint, it may or may not be connected



3. If a graph is vertex disjoint $\Rightarrow$ it will also a edge disjoint. [one way].

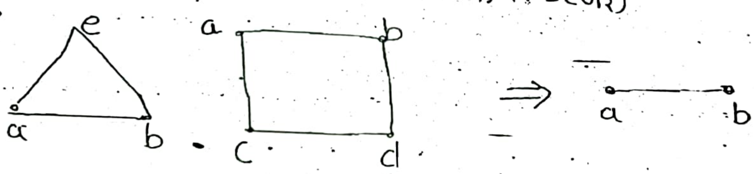
4. Let (n_1, e_1) is the cardinality for G_1 & (n_2, e_2) is the cardinality for G_2

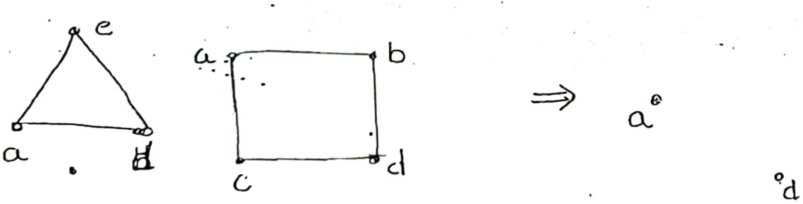
$$\boxed{\begin{aligned} n(G_1 \cup G_2) &= n(G_1) + n(G_2) - n(G_1 \cap G_2) \\ e(G_1 \cup G_2) &= e(G_1) + e(G_2) - e(G_1 \cap G_2) \end{aligned}}$$

$G_1 \cap G_2$:

$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$

$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$





Note-1. for $G_1 \cap G_2$ to be connected, G_1 & G_2 must be edge joint except if $|V_1 \cap V_2| = 1$

2. If two graph are vertex disjoint, $G_1 \cap G_2$ does not exist

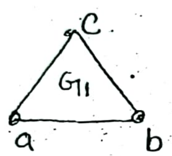
$$\boxed{G_1 \cap G_2 \text{ does not exist} \iff G_1 \,\&\, G_2 \text{ are vertex disjoint}}$$

3. $G_1 \cap G_2$ is connected iff :
   - $V(G_1 \cap G_2)$ has single vertex.

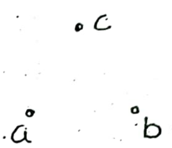
     [OR]
   - if $V(G_1 \cap G_2) \geq 1$ then Edge set must be joint.

3. <u>Complement of Graph</u>- it is done with respect to edge

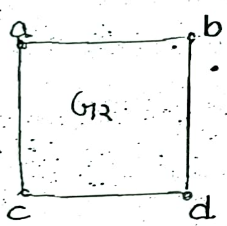
$$n(G) = n(\bar{G})$$
$$V(G) = V(\bar{G})$$



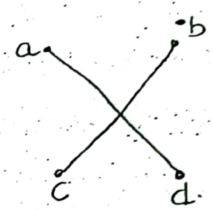
$\bar{G_1} =$



$$\boxed{\bar{G_1} = K_3 - G_1}$$



$\bar{G_2} =$



- Complement of a graph contains the edges which are not part of G but belong to corresponding complete graph.

$$\boxed{e(\bar{G_1}) = \frac{n(n-1)}{2} - e(G)}$$

- $\bar{G}(V', E') = K_{|V|} - G$

- $G \,\&\, \bar{G}$ are vertex joint & edge disjoint graph.

- if $G_1 \cup G_2 = K_n$, $G_1 \cap G_2 = \phi_n$ where n is the No of vertices in $G_1 \,\&\, G_2$

  Than $G_1 \,\&\, G_2$ are complement of each other

- Complement of Null graph = Complete graph

  $\bar{K_n} = \phi_n$

Ques- For a graph of $O(G) = 5$ & $e(G) = 7$
what is $O(\bar{G})$ & $e(\bar{G})$

- $O(\bar{G}) = 5$
- $e(\bar{G}) = {}^{n}C_2 - e(G) = {}^{5}C_2 - 7$
$= 3$

if the degree sequence of G (5, 5, 4, 4, 3, 3, 2)
what is degree sequence of $\bar{G}$?

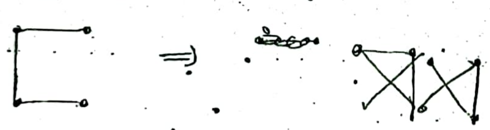
Ans- (5, 5, 4, 4, 3, 3, 2)
No of vertices = 7
- for $K_7 \Rightarrow$ Degree of each vertex = 6

$\therefore \bar{G} = K_7 - G$
$= (6, 6, 6, 6, 6, 6, 6) - (5, 5, 4, 4, 3, 3, 2)$
$= (1, 1, 2, 2, 3, 3, 4)$

Ques- if the deg. sequence of G $(d_1, d_2, d_3 \ldots d_n)$
what is the degree sequence of $\bar{G}$ in same order?
$\bar{G} = (n-1-d_1, n-1-d_2, n-1-d_3, \ldots, n-1-d_n)$

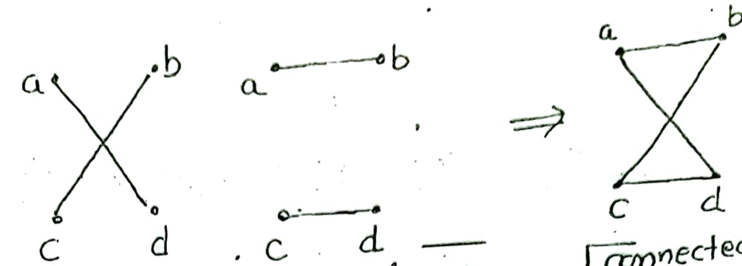
$$\boxed{\bar{G}\text{ (in same order)} = (n-1-d_n, n-1-d_{n-1}, \ldots, n-1-d_2, n-1-d_1)}$$

Note- 1. complement of a connected graph may or may not be connected



2. Complement of a disconnected graph is always connected.
3. Atleast one of G or $\bar{G}$ must be connected.

proof- if both are disconnected
than $G \cup \bar{G}$ can never be complete graph.
[It may or may not be connected but it can be complete graph]



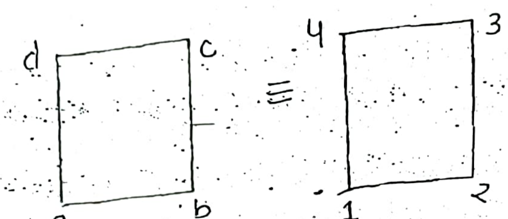
(Disconnected) (Disconnected)

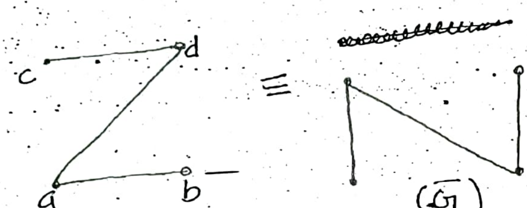
Connected but $G_1' \neq G_2$ because $G_1 \cup G_2$ is not complete graph

[G is disconnected $\Rightarrow \bar{G}$ is connected]

Self complementary Graph - Two graphs are self complementary if they are isomorphic to each other.

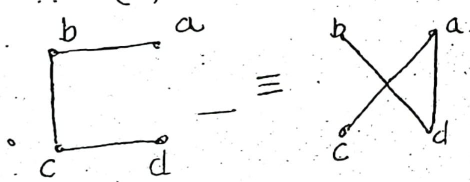
Isomorphic - Once the labels has been removed one should not be able to differentiate the Graph.





(G) ($\bar{G}$)

(Self complementary graphs)

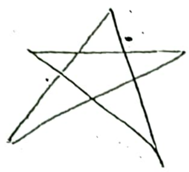


G ($\bar{G}$)

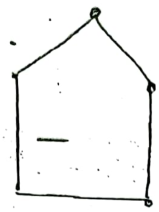
(Self complementary graphs)

If [G is self complementary than No of edges in $\bar{G}$ = No of edges in G $= \frac{n(n-1)}{4}$
No of vertices in $\bar{G}$ = No of vertices in G]

$\therefore \phi_n, K_n$ are not self complementary graph.



G $\xRightarrow{G'}$

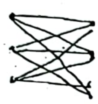


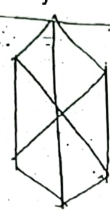
$\bar{G}$

$\Rightarrow$ Self complementary Graph
(both are isomorphic to each other)

Two graph G_1 & G_2 are self complementary iff
if G_1 & G_2 are isomorphic to each other

- $K_{3,3}$ is isomorphic to





complexity to check isomorphism $= O(n!)$

Property 1- If G is self complementary $\Rightarrow$ $n = 4x$ or $4x+1$ $\quad x \in Z$
ie $n \bmod 4 = 0$ or 1
n- No of vertex.

Property 2- if G is self complementary $\Rightarrow$ $\left[e = \frac{n(n-1)}{4} \right]$

Ques- A Graph G has 14 vertices then G is surely not self complementary

Ques A Graph with 9 vertices & 18 edges is given what you can say about G?
It may or may not be self complementary graph.

Ques- What is the no of vertices which does not form a sc graph?
(a) 13
(b) 12
(c) 15 ✓
(d) 16

Proofs:- ~~No of~~ in self complementary graph

No of edges in G = No of edges in $\bar{G}$ [due to isomorphism]

Also $e(G) = \frac{n(n-1)}{2} - e(\bar{G})$

$e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$

$2(e(G)) = \frac{n(n-1)}{2}$

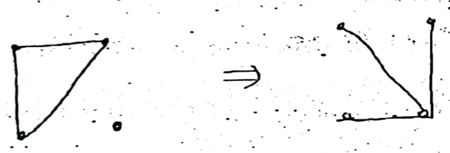
$$\boxed{e(G) = \frac{n(n-1)}{4}}$$

Also $n(n-1) = 4e$

$n = 4e$ or $n = 4e+1$

$\Rightarrow \boxed{n = 4x \text{ or } 4x+1}$

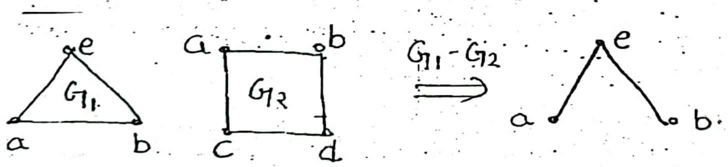
• It is one way theorem



[G] $\Rightarrow$ [$\bar{G}$]

$\Rightarrow$ not self complementary but $e = \frac{n(n-1)}{2}$ & $n = 4x$

4: **$G_1 - G_2$:** $G_1 - G_2 = G_1 \cap \bar{G_2}$

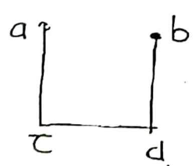


• $V(G_1 - G_2) = V(G_1)$

• $E(G_1 - G_2) = E(G_1) \cap E(\bar{G_2})$

$= E(G_1) - E(G_1 \cap G_2)$

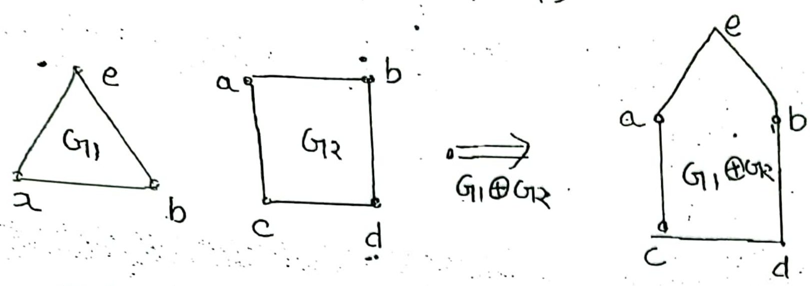
5. **$G_2 - G_1$**



• $\boxed{G_1 - G_2 \neq G_2 - G_1}$

$\therefore G_1 \oplus G_2 = (G_1 - G_2) \cup (G_2 - G_1)$

$\equiv (G_1 \cup G_2) - (G_1 \cap G_2)$



- It is also known as Ring-sum operator

$$\boxed{\begin{aligned} V(G_1 \oplus G_2) &= V(G_1) \cup V(G_2) \\ E(G_1 \oplus G_2) &= E(G_1) \oplus E(G_2) \end{aligned}}$$

Ques- if $|V_{G_1}| = 10$ $\quad |V_{G_2}| = 5$

$|E_{G_1}| = 8$ $\quad |E_{G_2}| = 3$ $\quad$ & $e|G_1 \cap G_2| = 2$

$V(G_1 \cap G_2) = 3$

What is $V(G_1 - G_2)$ & $E(G_1 - G_2)$?

$$\boxed{\begin{aligned} V(G_1 - G_2) &= V(G_1) = 10 \\ E(G_1 - G_2) &= 8 - 2 \\ &= 6 \end{aligned}}$$

- $V(G_2 - G_1) = 5$

$E(G_2 - G_1) = 1$

- $V(G_1 \oplus G_2)$ & $E(G_1 \oplus G_2)$?

$V(G_1 \oplus G_2) = V(G_1) \cup V(G_2)$

$= V(G_1) + V(G_2) - V(G_1 \cap G_2)$

$= 10 + 5 - 3$

$= 12$

$E(G_1 \oplus G_2) = E(G_1) \oplus E(G_2)$

$= E(G_1) + E(G_2) - 2E(G_1 \cap G_2)$

$= 8 + 3 - 2*2$

$= 11 - 4 = 7$

Isomorphism of Graph:

G_1 is isomorphic to G_2 ($G_1 \equiv G_2$) iff G_1 is equivalent to G_2 when labels are removed. ie underlying unlabelled graph are same.

$\boxed{G_1 \text{ is isomorphic to } G_2 \iff G_1 = G_2 \text{ when labels are removed.}}$

- G_1 & G_2 are isomorphic iff Adjacency matrix of G_1 $\equiv$ Adjacency Matrix of G_2 for some ordering of G_1 & G_2 (atleast one).

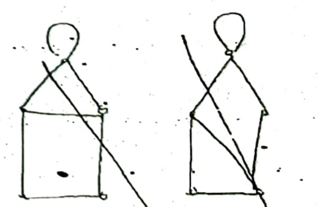
  Matrix are equivalent

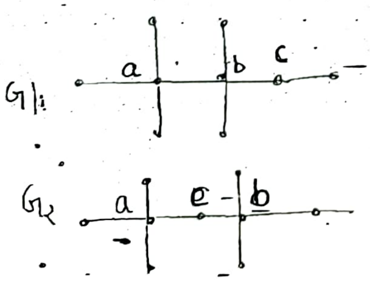
- the complexity of Algorithm to check for isomorphism is $O(n!)$ because it check the equivalence for every ordering of n.
- Matrix of two isomorphic graph can be same for more than one ordering of the graph.
- $\boxed{G_1 \text{ is not isomorphic to } G_2 \text{ iff } A_{G_1} \not\equiv A_{G_2}}$

Invariants for Isomorphism:

1. If G_1 & G_2 are isomorphic than [one way]

   1) $V(G_1) = V(G_2)$
   2) $e(G_1) = e(G_2)$
   3) degree sequence of G_1 = degree sequence of G_2
   4) No of cycles of any length must be same.
   5) for any two vertices u in G_1 & v in G_2 both should have ~~so~~ neighbours with the same quality.

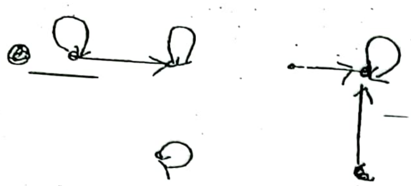




$V(G_1) = V(G_2)$
$e(G_1) = e(G_2)$
deg seq G_1 = deg seq G_2
No of cycles is also same

c does not satisfy the property for Neighbour.
Hence G_1 & G_2 is not isomorphic

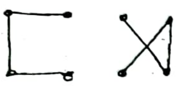
Qs $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ check for isomorphism?



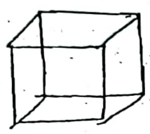
$\Rightarrow$ Not ~~an~~ isomorphic to each other.

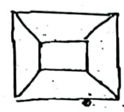
Whenever matrix are given, use graph to check for isomorphism.

tandard Isomorphism



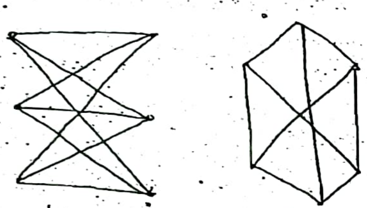
(i)





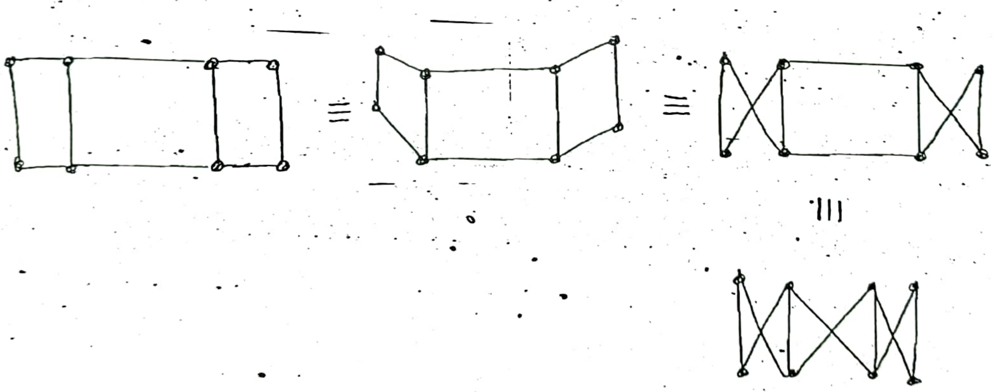
(ii)

)

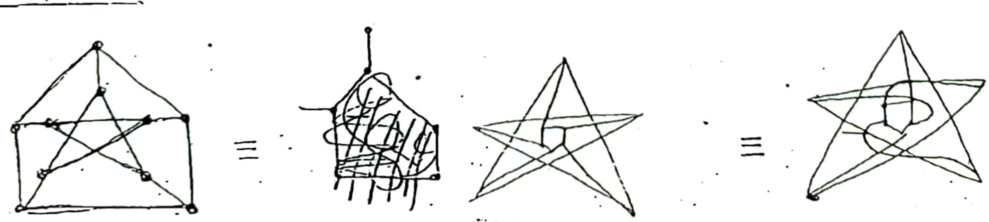


Note- $K_{3,3}$, K_5 are non planar graphs.

)

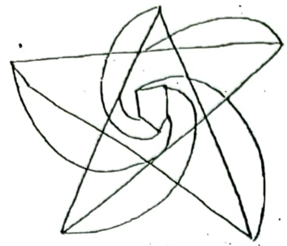


eter Son Graph-



[fully clockwise or anticlockwise]

Petersen Graph is isomorphic to



27-Sept:-

Isomorphism: $G_1 \equiv G_2$ iff there exist a bijection between $V(G_1)$ & $V(G_2)$ (isomorphic) in $f: V(G_1) \rightarrow V(G_2)$

such that if a is adjacent to b in G_1 then $f(a)$ must be adjacent to $f(b)$ in G_2.

[Adjacency preserving bijection]

Complexity $= O(n!)$

there will exist atleast one bijection for the given set.

Connectivity of Graph:

1. Connected & Disconnected Graph:

A Graph G is connected iff $\forall x, y \in G$, there exist a path between them.

graph is connected $\Leftrightarrow \forall x, y \in G \cdot \exists$ path $x \leftrightarrow y$

Smallest connected Graph

a

[Graph is disconnected $\Leftrightarrow \exists x, y \in G$ no path exist for $x \leftrightarrow y$]

Theorem 1. In any graph (either connected or disconnected) if there exist exactly 2 vertices of odd degree (x, y) then there exist a path b/w x & y.

proof- It is always true in connected graph.

in disconnected graph, if both are in same comp then there exist a path

but if in different comp, it is not possible because of the fact that no graph can have odd no of odd degree vertices.

Theorem 2- Let e be the no of edges
n- vertices
k- components

$$\boxed{\underbrace{n-k}_{\text{upto pseudo graph}} \leq e \leq \underbrace{\frac{(n-k+1)(n-k)}{2}}_{\text{for simple graph}}}$$

Let $n=10$, $k=3$
∴ Min. No of edges = 7
Max. No of edges = 28

When $k=1$,

$$\boxed{n-1 \leq e \leq \frac{n(n-1)}{2}}$$

$\left[e \propto \frac{1}{k} \quad e\uparrow\ k\downarrow \right]$

Ques- For a graph with 10 vertices & 6 edges what is the no of k?

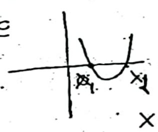
$n-k \leq e$
∴ $k \geq n-e$
$k \geq 10-6 \Rightarrow k \geq 4$ [atleast 4]

We can't say exactly 4, as it is allowing pseudo graph.

To Solve inequalities-
$x^2-5x+6 \geq 0$

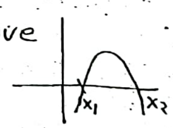
- Find the root
- In equn $ax^2+bx+c=0$

If $a=(+)ve$



$x \leq x_1$
$x \geq x_2$

If $a=(-)ve$



$x \geq x_1$
$x \leq x_2$

Ques- What is the max. no of edges in the disconnected graph with n vertices?

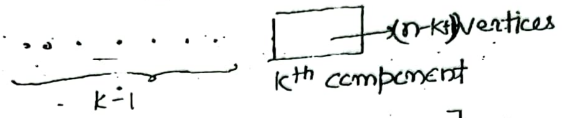
put $k=2$
∴ $e \leq \frac{(n-2+1)(n-2)}{2}$

$$\boxed{e \leq \frac{(n-1)(n-2)}{2}}$$ Max edges

∴ Max No of edges in simple graph with n vertices & still disconnected
$= n-2$ (atleast edges)
as soon as $n-1$ it can be connected [as done with the piegon hole principle]

$$n-2 \leq e \leq \frac{(n-1)(n-2)}{2}$$

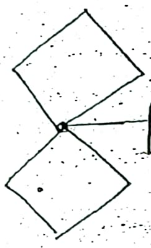
proof- for k components.
Max. edge will be there when imbalanced division is done.
ie for (k-1) components put 1 vertex only. ~~for~~ remaing. vertex are kept in a single components

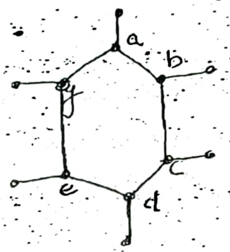


$$\left[\text{Max edges} = \frac{(n-k+1)(n-k)}{2}\right]$$

Min edges = $\underline{n-k}$

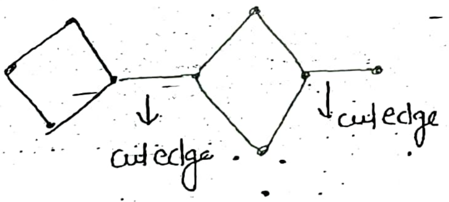
Cut Vertex- In connected graph, A vertex $v \in G$ is cut vertex if and only if removal of v disconnects the graph.
- It is also known as cut node [OR] articulation point





No of cut vertex = {a, b, c, d, e, f}
No of cut edge = {6}

Cut Edge- A edge $e \in G$ is cut edge if its removal disconnects the graph [In an connected graph]
- Also known as bridge.
- A pendant edge is always a cut edge.

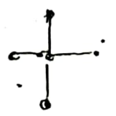


[$K_{1,n}$ → Star Graph.

$K_{1,1}$

$K_{1,2}$ 

$K_{1,3}$ 

$K_{1,4}$ ]

[No of cut edge = n
No of cut vertex = 1]

Normaly ~~K1~~
$K_{1,n}$ is always a tree.

Ques- How many cut edge & cut vertex in star graph with 10 vertices?

No of cut vertex = 1

No of cut edge = 9

Ques- Which graph can generate max no of components by removing 1 vertex?

Star graph (n-1) comp are generated.

Ques- How many max & min No of components can be generated by removing 1 vertex?

Min No of comp = 1

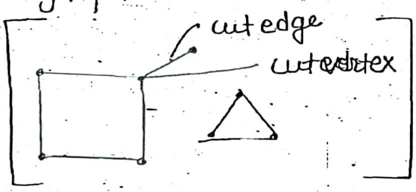
Max No of comp = n-1 (by star graph)

Diameter (Max path length in $K_{1,n}$), $n \geq 3 = 2$

$K_{1,1} = 1$

Note- for a connected or disconnected graph.

- cut vertex is a vertex in a graph whose removal increases the components of a graph.
- cut edge is one whose removal increases the components of a graph



Points-

- A tree with n vertices ($n \geq 2$) will have (n-1) cut edges
- A tree with n vertices ($n \geq 2$) will have i no of cut vertex where

i = No of internal nodes

$\boxed{i = n - l}$

↳ No. of cut vertex

No cut vertex exist in Cycle graph & complete graph and wheel graph

If e is a cut edge, than it must be included in every spanning tree of the graph.

3. If e is a cut edge with ~~minimum~~ max.weight, it will be included in every spanning tree of graph.

4. $\boxed{\text{A cut vertex 'v' is a vertex iff } \exists x,y \in G \text{ such that every path from } x \text{ to } y \text{ includes } v.}$

Formulae- In a rooted full m-ary tree

$\boxed{n = mi + 1}$

$i \rightarrow$ internal nodes

bcoz every internal node i will have m children & only 1 node is there which is not a child of any node i.e root.
$\therefore n = mi + 1$

Ques- How many cut vertex are there in full 3-ary tree with 22 vertices?

$n = mi + 1$ $\quad 22 = 3i + 1 \Rightarrow i = 7$

$n = i + l$ $\quad \boxed{l = 15}$ $\quad \boxed{\text{No of cut vertex} = i = 7}$

Ques- In a full k-ary tree with l nodes what is the No of internal nodes & leaf nodes
(I) (Lf)

$l = kI + 1$ $\Rightarrow$ $(k-1)I = Lf - 1$

$l = I + Lf$

$\left[I = \frac{l-1}{k}\right]$ $\quad \left[I = \frac{Lf-1}{k-1}\right]$ $\quad \left[Lf = L - \frac{Lf-1}{k-1}\right]$

$\left[Lf = l - \frac{l-1}{k}\right]$

Ques- Which of the following is not No of nodes in a full 3-ary tree?

(a) 21 (b) 31 (c) 22 (c) 13

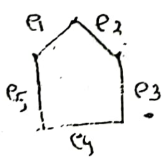
since $n = 3i + 1$

$21 - 1 = 3i$

i = not a fraction hence it is not allowed.

Cut Set- A minimal set of edges whose removal disconnects the graph.

- ~~more~~ than one edge exist.
- no subset of cutset should be able to disconnect the graph (proper)



cut set = $\{e_1, e_2\}$ ✓
$\{e_2, e_3\}$ ✓
$\{e_1, e_3\}$ ✓

$\{e_1, e_2, e_3\} \Rightarrow$ Not minimal
Not a cutset

- A cutset can't contain a cutset as a proper subset of it.

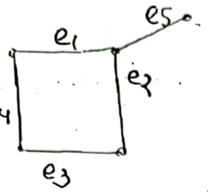
Which of the following is not a cut set?

(a) {e5}

~~(b)~~ {e1, e5} → proper subset {e5} is a cutset.

(c) {e1, e2}

(d) {e3, e4}



No of cutset-

1 edge cutset : {e5} → 1

2 edge cutset : No two edge cutset can involve 1 edge cutset elmt

$^4C_2$

$\boxed{\text{Total No of Cutset} = {}^4C_2 + 1 = 7}$

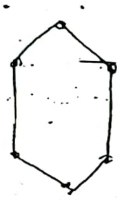
Note- A k edge cutset can never involve (k-1) edge cutset.

Vertex Connectivity & Edge Connectivity:

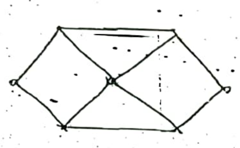
Vertex Connectivity- Min. No of vertex to be removed from the graph to disconnect it.

If a graph has cut vertex, VC = 1

If a graph doesnot have any cut vertex, VC ≥ 2



VC = 2



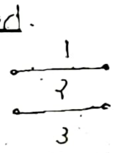
VC = 2

[To know the VC of a graph target the vertex with the minimum degree, i.e., $\boxed{VC \le \delta}$]

Edge Connectivity- Min. No of edge to be removed from graph to disconnect it.

- It will also be less than min. degree bcoz if all the edges of the node with min. degree is removed, graph will become disconnected.
- VC ≤ EC

bcoz, In worst case if n vertices are removed than n edges are removed.



[In worst case n vertex are needed to be removed. i.e, VC ≤ EC]

Therefore : $\boxed{VC \leq EC \leq \delta}$

What is the max VC & EC possible with graph of order 10 & size 16.

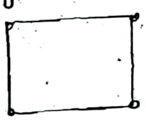
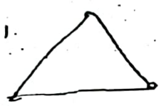
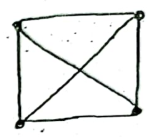
$\therefore \quad n = 10 \;\&\; e = 16$

$\therefore \quad \delta \leq \left\lfloor \frac{2e}{n} \right\rfloor = \delta \leq 3.2$

$\therefore \boxed{VC \,\&\, EC \leq 3} \quad \Rightarrow \quad \boxed{1 \leq VC, EC \leq 3}$

Theorem- A Graph G is separable if and only if VC = 1

Which of the following graph is separable?

(a) 
(b) 
(c) ✓ 
(d) 

A Graph G is not separable if & only if VC > 1

Note- A Graph is n-connected graph & k-line connected than
$$\begin{bmatrix} VC = n \\ EC = k \end{bmatrix}$$

Which of the following is true for a 3 connected & 4-line connected graph?

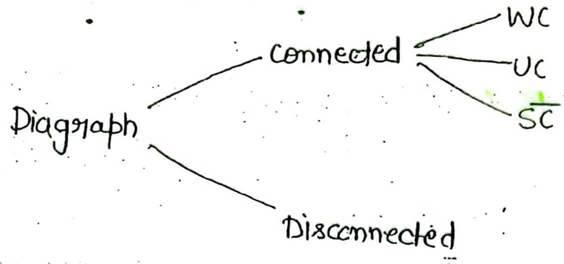
(a) Removal of any 3 Vertex can disconnected the graph. False
(b) Removal of some 4 edges can ______ True
(c) Removal of any 4 edges can ______ False
(d) ______ some 3 vertex ______ True
(e) Removal of 2 vertex cant disconnect the graph

- G, Separable Graph iff it is 1 connected.
- A Tree with $n \geq 3$ is 1 connected.
- A Tree with $n \geq 2$ is 1-line connected.
- For a cycle graph, $\boxed{\begin{matrix} VC = 2 \\ EC = 2 \end{matrix}}$
- For a complete graph, $\boxed{EC = n-1}$

VC = does not exist actually bcoz when (n-1) vertex are removed only 1 vertex left which is connected.

So we assume $\boxed{VC = n-1}$

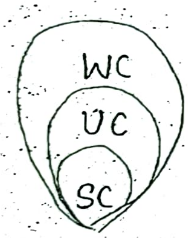
Strongly connected, weakly connected, unilateral connected, Disconnected.

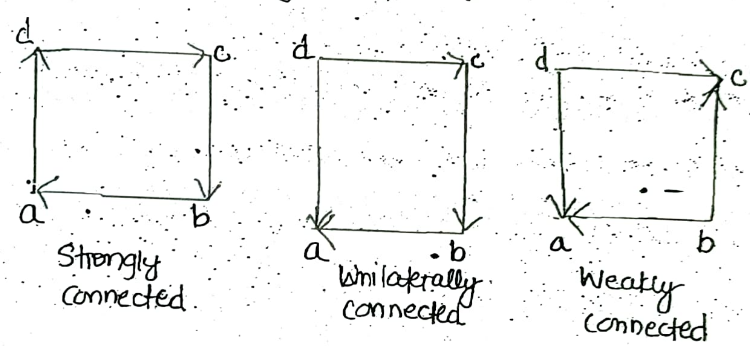


Unilateral connected- ∀ x, y ∈ G either a path from x to y exist or y to x exist.

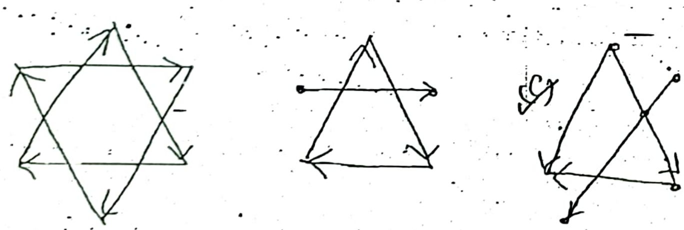
Strongly connected- ∀ x, y ∈ G path from x to y & y to x should exist.

Weakly connected- · No condition is there
· graph must have single component.

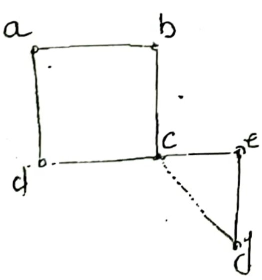




Which is connected?



Walk- Sequence of vertices such that b/w two successive vertices there should be an edge and edges should not repeat.



a b c e f → Walk
a b c e c → Not a walk (Same edge visited twice)
a b c e f c d → walk

Path- No vertices should repeat (but the starting vertex & ending vertex) may be same.

No ~~path~~ edge and should repeat

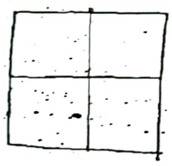
- A path is always a walk.
  - abcef → walk / Path
  - adcef → Walk / Path
  - abcda → path / walk.

Cycle- Every closed path is a cycle.

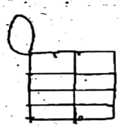
path → open path (Path)
path → Closed path (Cycle)

- Length of path is distinct no of vertices in it.
- abcefcd - open walk
- abcefcda - closed walk
- Self loops are cycle of length 1.
- Multiple edges are cycle of length 2.
- A simple graph have a cycle with min length = 3.

girth (G) ⇒ ~~Shortes~~ Length of shortest cycle in a graph.



⇒ ~~len~~ girth = 4



⇒ girth = 1

- girth of an Acyclic graph is 0
- girth of cycle graph is n.

Ques- What is the min. no of edges added to a tree with n vertices & n-1 edge to make it cyclic?

↳ 1 edge.

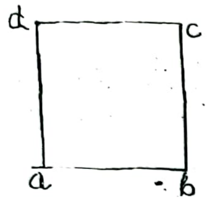
Every u-v walk contain u-v Path

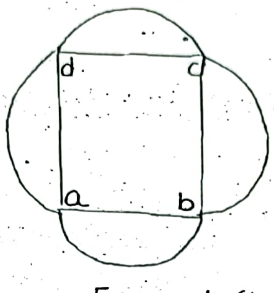
~~application of graph~~; application of Graph:-

Euler's Graph: G is Euler iff it is connected & $\forall v \in G$ degree(v)=even

G is Euler iff it contains a euler cycle.

Euler's Cycle- A closed walk which includes every edge exactly once.

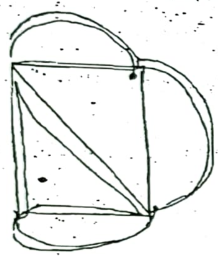




Euler cycle (but not a closed path)

xe degree must be even as if
le enter we should exit to complete
e cycle

In the euler's graph, no of vertices of odd degree = 0.



⇒ It contains Euler's path but not Euler's cycle

- A graph may contain more than one Euler's cycle
- A Null graph ϕ_n is not a euler graph
- A k-regular graph is Euler iff k is even & graph is connected.
- A complete graph K_n is Euler iff $n \geq 3$ & n is odd

hich of the following is euler graph?

✓ K_{51}

) K_{50}

) C_{60} ⇒ may be disconnected.

) 11-regular

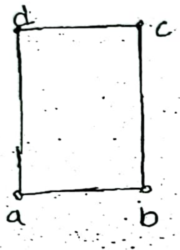
A cycle graph C_n is always Euler.

A wheel graph W_n is not a Euler graph because the vertices at outer edge ie (n-1) vertices will always have degree 3.

- A Q_n is ~~reg~~ Euler graph when n is even.
- A Bipartite graph may or may not be Euler [no comment can be made because it may be disconnected]
- A complete Bipartite graph $K_{m,n}$ is Euler if both m, n are even.

Euler path - an open walk which includes every edge.
- known as trail/euler line/euler walk.

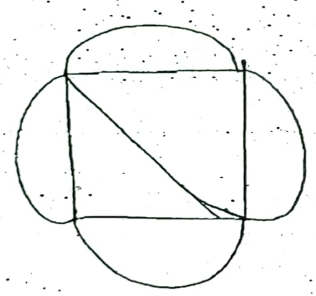
Unicursal graph - G is unicursal if and only if it contains a Euler path (open walk with every edge)



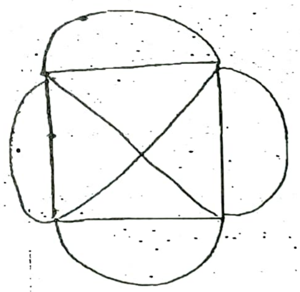
⇒ No euler path exist.

[If a graph contains Euler cycle, there will be no euler path.]

[A Graph G is unicursal iff it ~~contains~~ is connected & no of odd degree vertices = 2]



Euler path exist



No euler path & no euler cycle.

[The such euler path will start from one of these odd degree vertex & ends at other odd degree vertex.]

[if no(odd degree vertices) → 0 Euler Graph.
→ 2 Unicursal graph]

Traversal Graph - A Graph G is traceable/traversable iff it contains euler path or euler cycle.

ie [n(odd degree vertices) = 0 or 2] (for a connected graph)

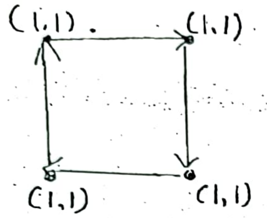
Advance Graph Theory

Directed Euler Graph-

A Directed Graph G is Euler graph iff it contains a directed Euler cycle

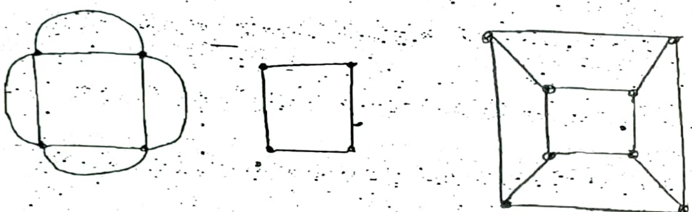
OR

iff it is unilaterally connected & $\forall_{v \in G}$ indeg(v) = outdeg(v)
(UC)
(as well as allowed SC)
bcoz SC is also UC



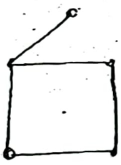
Hamiltonian Graph:

G is hamiltonian if & only if it contains a cycle (closed path)
(every vertex should be traversed ~~atleast~~ exactly once.)

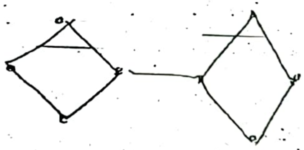


[edge may or may not be traversed but if traversed, should be done exactly once.

[Hamiltonian Cycle]



→ [Not a hamiltonian cycle] (Hamiltonian path exist)



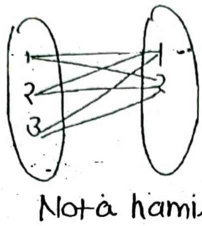
Not hamiltonian

- A Null $\emptyset_n$ is not hamiltonian graph
- A k-regular may or may not be hamiltonian
- A complete graph (K_n) with $n \geq 3$ is always hamiltonian graph.
  No of closed path = n! (Hamiltonian cycle)
- A cycle graph (C_n) is always a hamiltonian graph.
- A W_n (wheel graph) is always a hamiltonian ~~cycle~~ graph

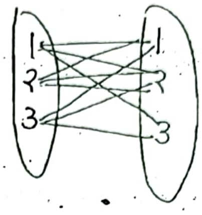
6.) A Q_n may or may not be hamiltonian graph
Q_2, Q_3 are hamiltonian graph.

7. A Bipartite graph may or may not be hamiltonian

8) A complete Bipartite graph is hamiltonian ($K_{m,n}$) when m & n are equal

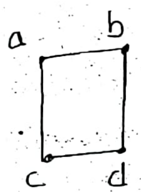


Not a hamiltonian graph



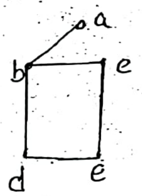
for [$m, n \geq 2$]

Hamiltonian path- ~~if each vertex is~~ open path including every vertex traversed once.



⇒ Hamiltonian path abcd

Theorem- if Hamiltonian cycle exist ⇒ Hamiltonian path will exist.



Ham path ⇒ abdec

Properties-

1. A Hamiltonian graph can never have a pendant edge.
[G is hamiltonian ⇒ No pendant edge]
Pendant edge ⇒ Not a hamiltonian graph.

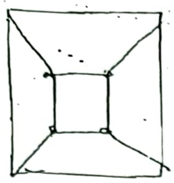
2. Dirac's Theorem- If a Graph is connected & $\forall v \in G$ $deg(v) \geq \frac{n}{2}$ than G is hamiltonian

Ques- If G is connected with degree seq. = {3, 3, 4, 4, 5, 5} than
— is it hamiltonian- True.
If G is connected with degree seq = {2, 3, 4, 5, 6, 7}
↳ [may or may not be hamiltonian]

ORE's theorem - if G is connected & $\forall u, v$ non-adjacent $\boxed{deg(u) + deg(v) \geq n} \Rightarrow$ G is hamiltonian

[one way]

Violation -

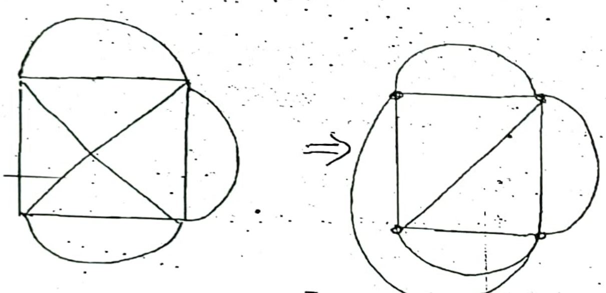


- No of Hamiltonian cycle of k_n - $n!$
- No of Hamiltonian cycle of k_n starting with specific vertex? $= (n-1)!$
- No of Hamiltonian cycle of k_n (unordered) $\Rightarrow \frac{n!}{n} = (n-1)!$
- Edge disjoint Hamiltonian cycle - $\left\lfloor \frac{n-1}{2} \right\rfloor$

? ways

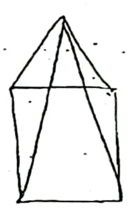
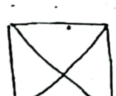
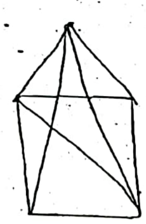
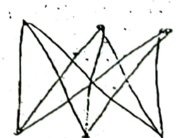
Planar Graphs:

A Graph G is planar iff $\exists$ planar representation (embedding) such that no edges are crossing each other.



- K_4 is a planar graph
- K_5 is not a planar graph

Which is non planar?

)  (b)  (c)  (d) ✓ 

$[K_{3,3}]$

Theorems:

Euler's formula for planar Graph - it cant be used to check planarity

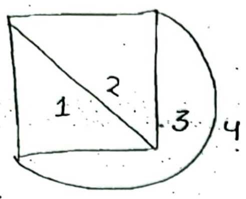
$\boxed{r = e - n + k + 1}$

e - no of edges
n - no of vertices
r → no of region (faces)
k → no of connected comp.

for a connected planar Graph,

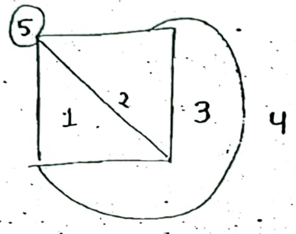
$\boxed{r = e - n + 2}$

- In all the regions, one will be open & other will be closed.



$n = 4$
$e = 6$

$\therefore$ regions $r = 6 - 4 + 2$
$= 4$



$n = 4$
$e = 7$
regions $= 7 - 4 + 2$
$= 5$

Ques- if a 3 regular graph of order 10. How many faces in planar embeeding for connected graph?

$\therefore e = \frac{3 \times 10}{2} = 15$

$n = 10$

$r = 15 - 10 + 2$
$= 7$

Ques- If a graph of order 10 & size 15 with 3 complonents is given. How many closed faces are there.

$r = 15 - 10 + 3 + 1$

$r = 9$

No of closed faces $= 9 - 1$
$= 8$

Ques In a planar graph every region is bounded by: ~~atleast~~ exactly 3 edges & order of graph is 10 & graph is connected. What is the No of regions

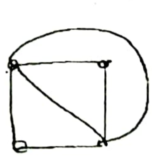
$r = e - n + 2$

$r = e - 8$

$r = \frac{3r}{2} - 8$

$\boxed{r = 16}$
$e = 24$

$e = \frac{3r}{2}$ [divide by 2 is done bcoz every edge is counted twice]



since every region is bounded is 3e then total region

$r = \frac{\text{total edges counted twice}}{3}$

ote- No of regions in a tree = 1

No of regions in bipartite graph (complete) = mn - (m+n)+2

Kuratowski's Theorem: A Graph G is planar if and only if it does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$.

Note- K_5 is smallest non-planar graph in terms of vertices & $K_{3,3}$ is smallest non-planar graph in terms of edges.

An Isomorphic graph is always homeomorphic

A homeomorphic graph may or may not be isomorphic.

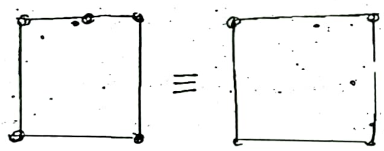
te- Any graph with vertices less than 5 is always planar.

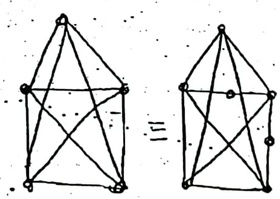
Any graph with edges less than 9 is always planar.

Homeomorphism- G_1 & G_2 are homeomorphic if G_1 can be obtained by fusion of two edges (Removal of a 2 degree vertex)

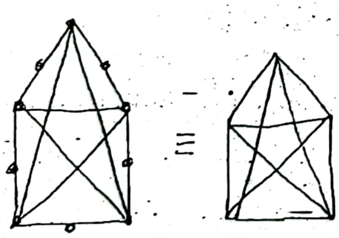
OR

Elementary Subdivision of edge (putting a new vertex on an edge)



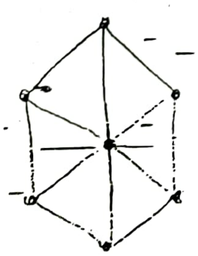


Homeomorphic to K_5

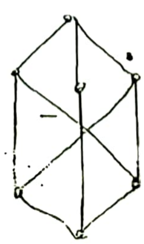


Homeomorphic to K_5

If a graph does not contain K_5, $K_{3,3}$ even after FED or ESD, then it will be planar (two way theorem)



→ planar [no edges are crossing]



⇒ equivalent (Homeomorphic) to $K_{3,3}$

⇒ Non-planar graph

3) In any connected simple planar graph

$$\boxed{e \leq 3n-6}$$

4) In any connected simple planar graph and No triangle, than

$$\boxed{e \leq 2n-4}$$

5) In a connected simple planar graph, minimum degree should be less than or equal to 5

$$\boxed{\delta \leq 5}$$

6) In a simple planar connected graph with no triangle

$$\boxed{\delta \leq 3}$$

proof- $r = e-n+2$ (for connected graph)

1. Also in Simple graph, every region is bounded by atleast 3 edges (as No self loop & multiple edge allowed)

$$r \leq \frac{2e}{3} \Rightarrow \left[e \geq \frac{3r}{2}\right]$$

$$\cancel{r = \frac{3r}{2} - n + 2}$$

Also $r = e-n+2$

$$e \geq \frac{3}{2}(e-n+2)$$

$$\boxed{3n-6 \geq e}$$

2. When no triangle is there, no region is bounded with 3 edges

$$e \geq \frac{4r}{2}$$

$$e \geq 2r$$

$$e \geq 2(e-n+2)$$

$$\boxed{e \leq 2n-4}$$

Ques Which of the following cannot be the edges in order to connected planar simple graph?

(a) 22
(b) 23
(c) 24
(d) 26 ✓

$e \leq 3 \times 10 - 6 \leq 24$

$\therefore$ Min No of edges for it to be non planar = 25

In CPS graph $\delta \leq 5$

: As we know that $\delta \leq \frac{2e}{n}$

in CPS $e \leq (3n-6)$

$\therefore \delta \leq \left\lfloor 6 - \frac{12}{n} \right\rfloor$

↳ the max value here can be 5.9999...

$\boxed{\delta \leq 5}$

1. In CPS with No triangle, $\delta \leq 3$

$\delta \leq \frac{2e}{n}$

$\delta \leq \frac{4n-8}{n}$

$\delta \leq \left\lfloor 4 - \frac{8}{n} \right\rfloor$

$\boxed{\delta \leq 3}$ $\therefore$ [Max value of min. degree of CPS with No triangle = 3]

Trees:

1. A Graph G is tree iff G is connected & Acyclic.

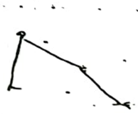
Graph with n vertices is a tree if and only if

- Connected and Acyclic
- Connected & $(n-1)$ edges.
- minimally connected graph
- Acyclic & $(n-1)$ edges
- $\forall x, y \in G$ there exist exactly one path b/w x & y

[Since path exist ⇒ ~~cyclic~~ connected
exactly one path ⇒ Acyclic]

C & A ⇒ n-1 edges
n-1 & A ⇒ connected
C & n-1 ⇒ Acyclic

[If a graph is minimally connected, every edge is needed to keep graph connected]



- Multiple path exist only in cyclic graph.

Every tree with 2 or more vertices is bichromatic & bipartite. $n \geq 2$

Tree is a 1-line connected graph $n \geq 2$

Tree with $n \geq 3$ is 1 connected graph.

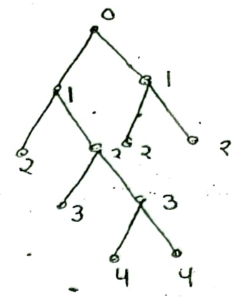
e- In a tree, if an edge is added b/w any two vertices, than exactly one cycle will be generated & created cycle is known as fundamental cycle.

$\therefore$ [True for tree only]

Ques- How many fundamental ~~ver~~ cycle possible in tree of 10 vertices?

$^{10}C_2$

Rooted Tree-



Depth (vertex): Distance b/w the root & vertex.

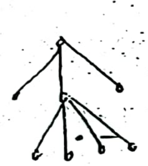
Height (Vertex): Maximum depth of any vertex in the tree.

Level (Tree) = No of Levels in tree
= height + 1

Level - n - vertices = Vertices present at level n in the tree.

m-ary tree-

A rooted tree where
$0 \leq$ no of children at each node $\leq m$



$[m=4]$

$n = i + l$

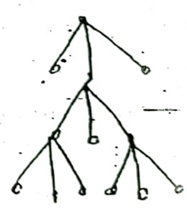
$i \downarrow$ internal node

$l \rightarrow$ leaf nodes

full m-ary tree-

A rooted tree where

$\boxed{\text{No of children at each node} = 0 \text{ or } m}$

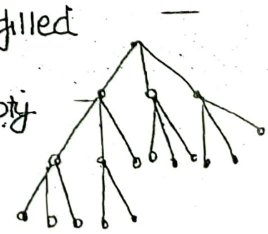


full 3 ary tree

- $n = i + l$
- $n = mi + 1$

Complete m-ary tree :

- every level should be fully filled except last level.
- last level is allowed to be empty or left justified.



A complete m-ary tree may not be full m-ary tree & vice versa.

For a m-ary tree, of height h,

$$\boxed{\begin{array}{l} l \leq m^h \\ n \leq \dfrac{m^{h+1}-1}{m-1} \end{array}}$$

l - leaf nodes

n - total nodes

h - height of tree

$$\text{Total No of nodes} = m^0 + m^1 + \dots + m^n \quad [\text{on Max}]$$
$$= 1 + m + m^2 + \dots + m^h$$
$$= \frac{1(m^{h+1}-1)}{m-1}$$

For a binary tree– $\boxed{\begin{array}{l} l \leq 2^h \\ n \leq 2^{h+1}-1 \end{array}}$

Also no of internal nodes $i \leq \frac{m^{h+1}-1}{m-1} - m^h$

$$\leq \frac{m^{h+1}-1 - m^h(m-1)}{m-1}$$

$$\boxed{\text{No of internal nodes} \leq \frac{m^h-1}{m-1}}$$

Minimum No of nodes– $n \geq h+1$

$i \geq h$

$$\boxed{\begin{array}{l} h+1 \leq n \leq \frac{m^{h+1}-1}{m-1} \\ h \leq i \leq \frac{m^h-1}{m-1} \end{array}}$$

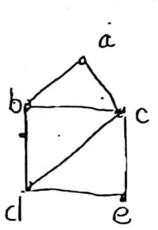
height of tree with l leaf nodes, $h = \lceil \log_m l \rceil$

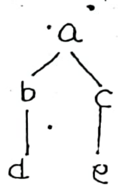
Spanning Tree of a Graph– Every connected graph has a spanning tree.

Spanning tree– A tree subgraph $T(G)$ of G iff

1) T is a tree

2) T includes every ~~subgraph~~ vertex of G..



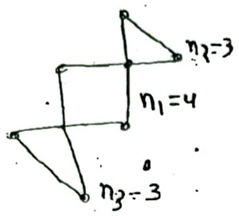


Removing both common edge + 1 common edge keep + 2nd common edge keep + both keep 5+6+6+4

$= 5 + 2*3 + 3*2 + 2*2$

$= 19.$

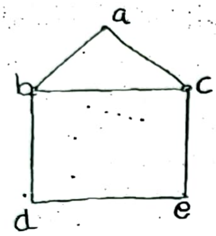
No of Spanning tree-



If the graph is cycle disjoint graph

$\therefore$ No of spanning tree $= n_1 \times n_2 * n_3$

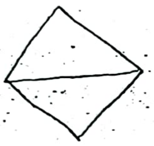
$= 36$



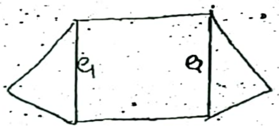
$\Rightarrow$ by removing the common edge + by keeping the common edge

$= 5 + 2*3$

$= 11$



$= 4 + 2*2 = 8$



= e_1 & e_2 both not taken + e_1 taken & e_2 not + e_2 taken & e_1 not + e_1 & e_2 both taken

$= 6 + 2*4 + 2*4 + 2*2*3$

$= 6+8+8+8 = 30$

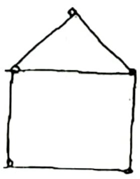
In weighted tree, spanning tree with minimum weight is min spanning tree, may or may not be unique.

Rank & Nullity of Graph

~~If~~ for a graph with n vertices, e edges & k components,

$$\boxed{\begin{aligned} Rank(G) &= n-k \\ nullity(G) &= e-n+k \end{aligned}}$$

$$\boxed{Nullity(G) = \text{No of edges} - Rank(G)}$$



$n=5$
$e=6$
$k=1$

$$\boxed{Rank(G) + Nullity(G) = \text{No of edges}}$$

$\therefore Rank(G) = 4$

$Nullity(G) = 2$

Rank (G) $\Rightarrow$ No of edges in the spanning tree of graph (for a connected graph)

$= n-1$

$\Rightarrow$ No of edges in the spanning forest of disconnected graph ɪt n be the total vertices as

$n_1 \quad n_2 \quad n_3$ in the componends upto k components.

$$\therefore \boxed{\begin{aligned} \text{No of edges} &= n_1 - 1 + n_2 - 1 + n_3 - 1 + \dots + n_k - 1 \\ &= n_1 + n_2 + n_3 + \dots + n_k - k \\ &= n - k \end{aligned}}$$

Nullity - No of edges needed to be removed to convert the given graph into spanning tree/forest

- It is also known as cyclomatic complexity of the graph.
  (time for ~~brak~~ break the cycle / No of cycle which are broken)

ranch Set - Set of edges which are part of spanning tree.

hord Set - Set of edges which are not taken to convert graph into the spanning tree.

$$\boxed{\begin{aligned} |\text{branch set}| &= \text{Rank}(G) \\ |\text{Chord set}| &= \text{Nullity}(G) \end{aligned}}$$

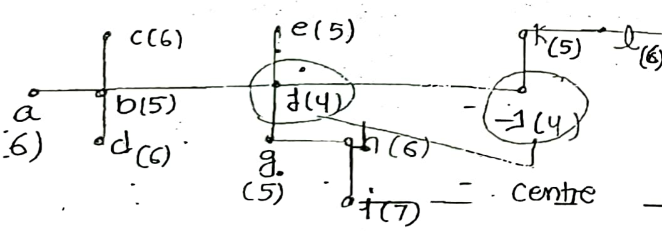
dius & DiaMeter of Tree:

ccentricity (V): Distance of the farthest vertex from v

entre (Tree): A vertex with minimum eccentricity is the centre of tree.
A tree can have either 1 center or 2 center
(Unicentric Tree) (Bicentric Tree)

adius (Tree): Minimum value of eccentricity
≡ eccentricity of ~~Radius~~ Center

iameter (Tree): Maximum eccentricity in the tree.



bicentric tree

centre = {d, j}

Radius = 4

Diameter = 7.

- A tree will be unicentric or bicentric only.
- Centre is also obtained by repeatedly deletion of leaf nodes from the tree.
- When all the leaf node removed, eccentricity of each & every node decreases by 1.
- At last only 2 points or 1 point (node) is left, which corresponds to center.

if 3 nodes are there

⇓

Algo will be repeated again & so on.

• (left with 1)

⇓

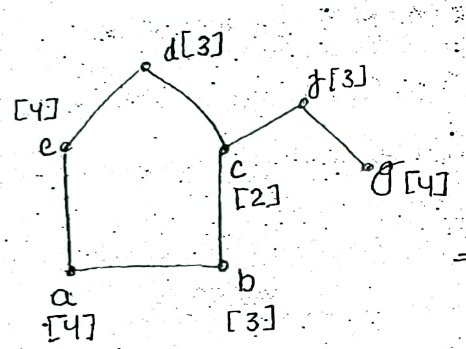
(left with 2)

27-Sept-2017

Enumeration of Graphs:

for a graph-

Eccentricity - shortest path to the farthest node.

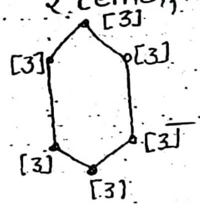


Centre = c

Radius = 3

Diameter = 4

Note- it is possible to have more than 2 centers in a graph.



⇒ $\boxed{\text{center} = 6 \text{ are possible.}}$

- No of Labelled graph possible with n vertices? $2^{n(n-1)/2}$
- No of Labelled graph with n vertices & e edges? ${}^{\frac{n(n-1)}{2}}C_e$
- No of Simple Labelled graph with n vertices? n^{n-2}
- No of Labelled trees possible with n vertices (k_n) n^{n-2} } Cayley formula.
- No of spanning trees possible with n vertices
- No of rooted labelled tree with n vertices n^{n-1}
- No of labelled subgraph for a k_n graph (labelled) $\sum_{r=1}^{n} {}^{n}C_{r}\, 2^{\frac{r(r-1)}{2}}$

E 1. with n vertices

Max No of edges = $\frac{n(n-1)}{2}$

∴ possible graphs

$= 0\text{ edge} + 1\text{ edge} + \dots \frac{n(n-1)}{2}\text{ edge}$

$= {}^{\frac{n(n-1)}{2}}C_0 + {}^{\frac{n(n-1)}{2}}C_1 + \dots + {}^{\frac{n(n-1)}{2}}C_{\frac{n(n-1)}{2}}$

$= 2^{\frac{n(n-1)}{2}}$

2.) with n vertices & e edges

Max No of edges = $\frac{n(n-1)}{2}$

∴ $\left[\text{ways to have edges} = e \Rightarrow {}^{\frac{n(n-1)}{2}}C_e \right]$

Ques- What is the No of graph possible with n vertices & atleast $\frac{n(n-1)}{4}$ edges?

$\Rightarrow {}^{\frac{n(n-1)}{2}}C_{\frac{n(n-1)}{4}} + {}^{\frac{n(n-1)}{2}}C_{\frac{n(n-1)}{4}+1} + \dots + {}^{\frac{n(n-1)}{2}}C_{\frac{n(n-1)}{2}}$

$\Rightarrow \sum_{K=\frac{n(n-1)}{4}}^{\frac{n(n-1)}{2}} {}^{\frac{n(n-1)}{2}}C_K$

$\Rightarrow \boxed{\sum_{K=0}^{n(n-1)/4} {}^{\frac{n(n-1)}{2}}C_0}$

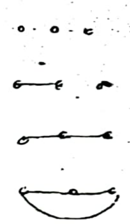
No of unlabelled graph with 3 vertices

0 edge -

1 edge -

2 edge -

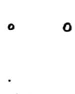
3 edge -



{ No of graph possible = 4 }

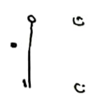
n vertices = 4

0 edge



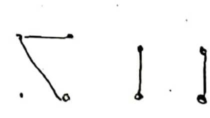
= 1

1 edge



= 1

2 edge



= 2

3 edge- 

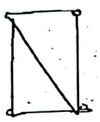
8 4 3 4

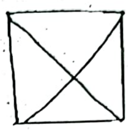
(it is isomorphic to ) = 3

4 edge- 

= 2

[No of 4 edges graph = No of 2 edge graph (because of complement property]

5 edge-  = 1

6 edge 

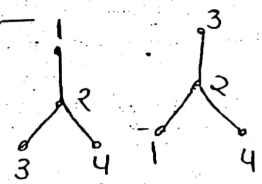
= 1

total possible- 1+1+2+3+1+1+2
= 9+2 = 11

No of rooted tree- [No of tree * n]
↳ n elements can become the root

$= n^{n-2} * n$

$= n^{n-1}$



No of subgraphs of kn labelled graph-

= graphs with 1 vertices + graph with 2 vertices + . . . + graphs with n vertices

$= {}^{n}C_1 * 2^{1(1-1)/2} + {}^{n}C_2 \cdot 2^{2(2-1)/2} + \dots + {}^{n}C_n \cdot 2^{n(n-1)/2}$

$$\boxed{= \sum_{r=1}^{n} {}^{n}C_r \cdot 2^{\frac{r(r-1)}{2}}}$$

$$\boxed{= \sum_{r=1}^{n} {}^{n}C_r \sum_{e=0}^{\frac{r(r-1)}{2}} {}^{\frac{r(r-1)}{2}}C_e}$$

→ many subgraph possible with 3 edges in 5 vertices graph
edges are possible with n=4 & n=5 & n=3

total subgraph = 3 vertex 3 edge + 4 vertex 3 edge + 5 vertex 3 edge

$$= \left[{}^5C_3 \cdot {}^{\frac{3(3-1)}{2}}C_3 + {}^5C_4 \, {}^{\frac{4(4-1)}{2}}C_3 + {}^5C_5 \, {}^{\frac{5(5-1)}{2}}C_3 \right]$$

raph No:

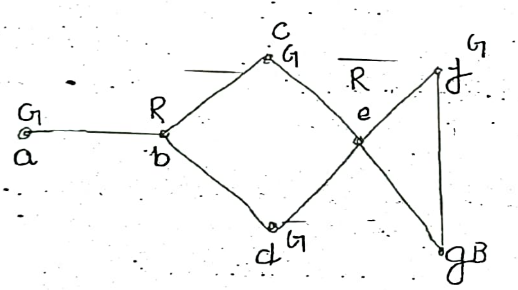
Chromatic No (K_G): Minimum No of colors required for proper coloring of graph.
(No two adjacent vertex have same color)

Independence No (β_G) (vertices): Size of the largest maximal independent set.

Domination No (γ_G) (vertices): Size of the smallest minimal dominating set.

Matching No (edge) M_G: Size of largest maximal matching.

Covering No edges ϵ_G: Size of smallest minimal covering



chromatic No = 3

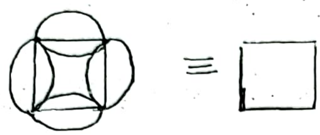
eorem - 1. chromatic No of a graph is atleast equal to size of largest clique.

$\boxed{K_G \geq \text{size of largest clique}}$ if graph contain K clique $\boxed{K_G \geq K}$

∴ In the graph above, biggest clique is K_3.
$K_G \geq 3$

Note - 1. There is no meaning of chromatic No with pseudo graph.
[because a node with self loop can never get colored]

2. Chromatic No of a multigraph is equal to chromatic No of its equivalent simple graph.
[Removal of multiple edges]

 $\Rightarrow K_G = 2$

 $\Rightarrow \boxed{K_G = 3}$

3. Chromatic No of a complete graph with n vertices is n.

4. chromatic No of a graph with atleast 1 edge is atleast 2.

5. A Graph is Null if and only if $K(G) = 1$

6. Chromatic No of a planar graph is atmost 4 [4-color conjecture]
[K(Planar graph) ≤ 4]

7. for any graph,

$\boxed{K_G \leq 1 + \Delta \leq n.}$ [Δ - maximum degree]

there exist a node with max. degree $= \Delta$
then each node will have deg less than that or in worst case, it is equal to that.
than a one more color is needed to color the node with degree Δ.

8.) Independence No, $\boxed{\beta_G \geq \frac{n}{K_G}}$ [using piegon hole principle]

Independent Set - Set of the vertices which are not adjacent ~~connected~~ to each other. [none of them are adjacent]

if $n = 10$, $K_G = 2$ $\Rightarrow \beta_G \geq 5$

Because - atleast one of the set will have $\lceil n/K_G \rceil$ elements
because of K_G colors used for coloring
(the nodes with same color are not connected ie adjacent to eachother)

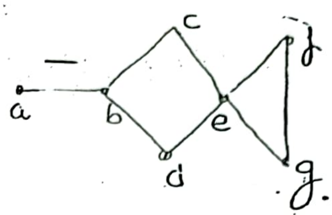
which of the following is a independent set?

i. {a}

ii. {b}

iii. {a, c, d, f}

v {a, c, d, e}



• A single vertex is always an independent set.

n independent set is maximal if even one of the remaining vertex is added should violate the condn for independent set.

{b, e} – Maximal Independent Set

{a, c, d, f} → Largest maximal Independent set.

Largest maximal independent set may or may not be unique but independence No is unique.

To draw independent set which is maximal.

1. Always start with minimal degree vertex.
2. Starting adding the vertices by checking adjacency

{a, c, d, f}

{a, c, d, g}

Independence No = |Largest maximal independence set|

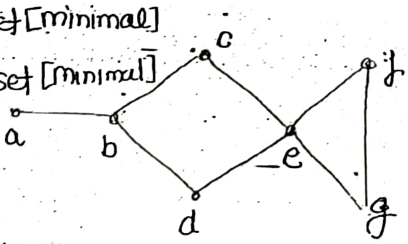
= 4

Domination No – Smallest minimal dominating set.

Dominating set: Set of vertex from which the whole graph can be covered in the single move.

{e, b} → Dominating set [minimal]

{a, c, d, f} → Dominating set [minimal]



Theorem – [Maximal independent set ⇒ Dominating set]

– A minimal dominating set may or may not be independent set.

{e, b} – smallest minimal dominating set.

To draw a smallest minimal dominating set -

- take the vertices ~~with~~ with the maximum degree.
- further add the vertices which are left uncovered.

$\therefore$ Domination No $\gamma_G = 2$

Theorem - Domination No $\leq$ Independence No

$$\boxed{\gamma_G \leq \beta_G}$$

Since a max. independent set is always a dominating set

$\therefore$ Independent No $\geq$ Dominating No.

4. Matching No : - for edges

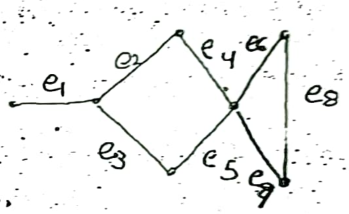
- size of largest maximal matching.

Matching - Set of edges none of which are adjacent to each other.

$\{e_1\}$

$\{e_1, e_8\}$

$\{e_1, e_4, e_8\}$



To obtain Largest maximal matching.

- Start with the edges with less no. of adjacency
- & continue.

$e_1 - 2$, $e_6 = 4$

$e_2 \to 3$, $e_7 = 4$

$e_3 - 3$, $e_8 = 8$

$e_4 - 4$

$e_5 \to 4$

$\{e_1, e_8, e_4\}$ } Largest Maximal Matching

$\{e_1, e_5, e_8\}$

- it may or may not be unique.
- Matching No is always Unique.

$\Rightarrow$ $\boxed{\text{Matching No} = |\text{Matching}| = 3}$

~~Theo~~ 5. Covering No : Size of smallest minimal cover.
- for edges.

Covering - Set of edges which covers all the vertices.

$\{e_1, e_4, e_6\} \to$ False Not a covering

$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ - Covering

$\therefore \{e_1, e_4, e_8\} \to$ Minimal covering (smallest)

$\{e_2, e_1, e_3, e_7, e_6\}$ - minimal covering but not smallest.

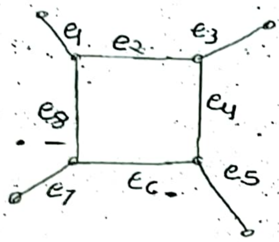
Theorem- The covering of a graph is atleast $\lceil n/2 \rceil$

$|cover| \geq \lceil n/2 \rceil$

proof- Covering involves edges an edge can covert atmost 2 edges $\therefore$ to covert n vertices atleast $\lceil n/2 \rceil$ need to be added.

Theorem-
- A pendant edge/cut edge is always part of every cover.
- A spanning tree will always include pendant edge.

Procedure-
1. firstly includes all the pendant edge
2. Now try to include all the edge which added new vertices to it.



$= \{e_1, e_3, e_5, e_7\}$

$\therefore$ $\boxed{\text{Covering No} = |\{e_1, e_3, e_5, e_7\}| = 4}$

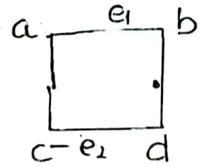
Theorem- A matching need not be covering & a covering need not be matching

Pf- bcoz in ~~matching~~ covering we can include adjacent edges (in case of pendant edge)

~~a covering need not be covering~~

A matching need not be covering bcoz it does not need to include all the vertex.

Theorem- A perfect matching $\Rightarrow$ Matching + covering

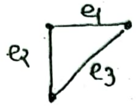


Match = $\{e_1, e_2\}$

Covering = $\{e_1, e_2\}$

- Perfect matching is possible for a graph, when no of vertices is even.

In case of odd vertices

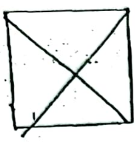


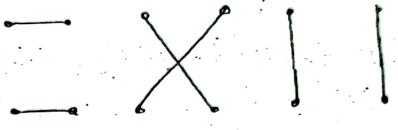
Matching = {e1}
cover = {e1, e2}

If a graph has even vertices
→ it may or may not have perfect matching.

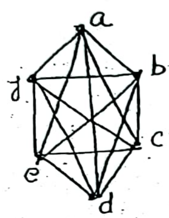
Perfect Matching for K_{2n}:

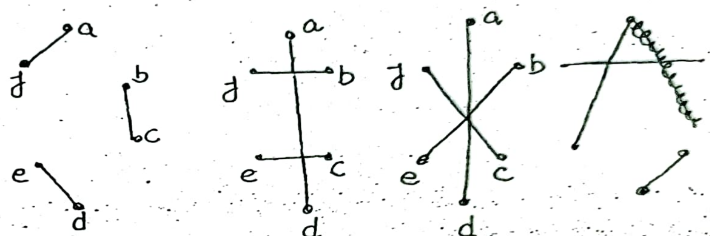
K_4 -





K_6.





2n → element getting divided into n groups of 2 elements each

K_6 perfect matching $= \frac{6!}{(2!)^3 3!}$

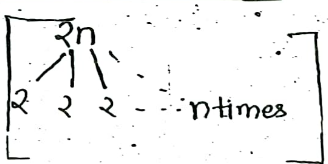
$= \frac{\cancel{6}^{3} \times 5 \times \cancel{4}}{\cancel{2} \times \cancel{2} \times \cancel{2}}$

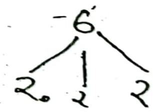
$= 15$

$$\boxed{\text{perfect matching for } K_{2n} = \frac{(2n)!}{(2!)^n n!}}$$

$K_4 = \frac{4!}{2!\,2!\,2!} = \frac{\cancel{4} \times 3 \times \cancel{2}}{\cancel{2} \times \cancel{2} \times \cancel{2}}$

$= 3$

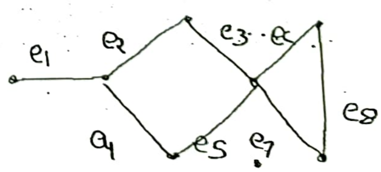


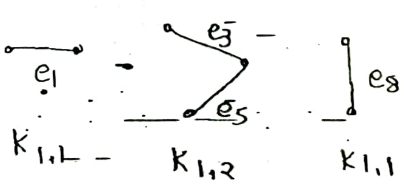


Theorem- A cover is minimal iff only if every component is star graph $K_{1,n}$

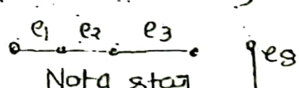
{e1, e3, e5, e8}

∴ components are





{e1, e2, e3, e8}



Not a star graph

⇒ Hence it is not minimal

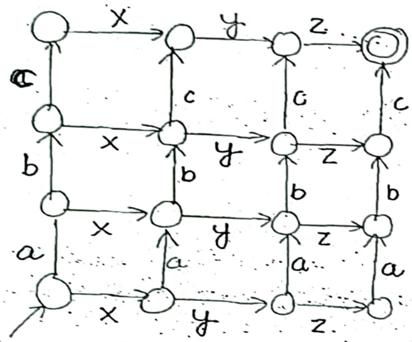
$\forall x \, \forall y \, (\text{person}(y) \Rightarrow (\text{respects}(y, x) \Rightarrow \text{King}(x)))$ preposition

~~if a~~ All person y choose all king when respect every x.

getting true for only single y:

if only a person y respect every some person $\Rightarrow$ it will become king

Grid Problem:



$^6C_3 \cdot {^3C_3}$

$\downarrow$ to choose 3 Right move

$\uparrow$ to fill three up move

x y z a b c

filling will always be unique but the place can change

seq will always be

$\begin{bmatrix} x - y - z \\ \& \\ a - b - c \end{bmatrix}$

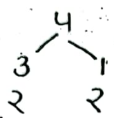
No of equivalence relation = No of partition of a given set

let $A = \{1, 2, 3, 4\}$

1-block partition

$\{1, 2, 3, 4\} \rightarrow 1$ way

2 block partition-



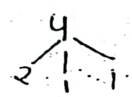
$\frac{4!}{3!\,1!} + \frac{4!}{2!\,2!\,2!}$

$= 4 + 3$

$= 7$ ways

$\{\{1,2\}\{3,4\}\}$
$\{\{1,3\}\{2,4\}\}$
$\{\{1,4\}\{2,3\}\}$
$\{\{1,2,3\}\{4\}\}$
$\{\{1,3,4\}\{2\}\}$
$\{\{2,3,4\}\{1\}\}$
$\{\{1,2,4\}\{3\}\}$

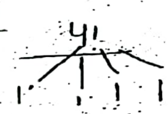
3-block partition-



$\frac{4!}{2!\,(1!)^2\,2!} = \frac{4!}{4} = 6$ ways

$\{\{1,2\}\{3\}\{4\}\}$ $\{\{1,4\}\{2\}\{3\}\}$
$\{\{2,3\}\{1\}\{4\}\}$ $\{\{1,3\}\{2\}\{4\}\}$
$\{\{3,4\}\{1\}\{2\}\}$ $\{\{2,4\}\{1\}\{3\}\}$

4-block partition



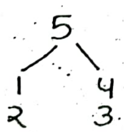
$\Rightarrow \frac{4!}{(1!)^4\,4!} = 1$ way $\{\{1\}\{2\}\{3\}\{4\}\}$

$\therefore$ Total ways = 15

$A = \{1, 2, 3, 4, 5\}$

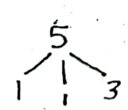
1-block partition- 1 way

2 block partition-



$\frac{5!}{4!\,1!} + \frac{5!}{3!\,2!} = 5 + 10 = 15$

3- block partition



$\frac{5!}{3!\,2!\,1!\,1!} + \frac{5!}{2!\,2!\,2!\,1!}$

$= 10 + 15$

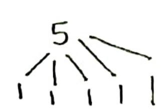
$= 25$

4. block partition-

$\frac{5!}{2!\,3!} = 10$

Ways $= 1 + 15 + 25 + 10 + 1$
$= 52$ ways

5 block partition-



$= \frac{5!}{5!} = 1$

Inrement- $\Pi_1 \, R \, \Pi_2$ iff Π_1 refines Π_2.

Π_1 refines Π_2 iff every block of $\Pi_1 \subseteq$ some block in Π_2

Ques 39- $a = \{\{1,2,3\}\{4\}\{5\}\}$

$b = \{\{1\}\{2,3\}\{4,5\}\}$

a does not refines b

b does not refines a

~~b ≤ a~~

$$\boxed{\text{if } \Pi_1 \text{ refines } \Pi_2 \text{ than Join} = \Pi_2,\ \text{Meet} = \Pi_1}$$

~~$\Pi_1 = \{1 \ 2$~~ $\Pi_1 = \overline{1234}$

$\Pi_2 = \overline{12}\ \overline{34}$

$\Pi_3 = \overline{13}\ \overline{24}$

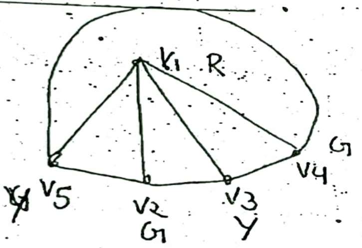
$\Pi_4 = \overline{1}\ \overline{2}\ \overline{3}\ \overline{4}$

$\Rightarrow$

$$\begin{tikzcd}
 & \Pi_1 \arrow[dl, dash] \arrow[dr, dash] & \\
\Pi_2 \arrow[dr, dash] & & \Pi_3 \arrow[dl, dash] \\
 & \Pi_4 &
\end{tikzcd}$$

• Diagonal matrix are commutative & form abelian group.

Chromatic Partitions-



Chromatic partitions

$= \{[V_1][V_2, V_4][V_5, V_3]\}$

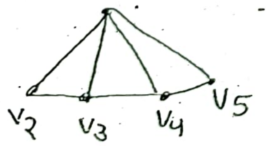
$= \{\{V_1\}\ \{V_2, V_4\}$

V2 → can only go with V4

V5 → can only go with V3

V1 → can only go alone

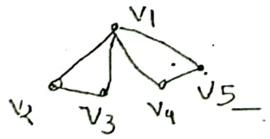
∴ No of chromatic partition = 1



$\{V_1\}\ \{V_2, V_4\}\ \{V_3, V_5\}$.

$\{V_1\}\ \{V_2, V_5\}\ \{V_3, -\}\ \{V_4\}$

∴ 1 partition



$\{V_1\}\ \{V_2 - V_4\}\ \{V_3 = V_5\}$

$\{V_1\}\ \{V_2, V_5\}\ \{V_3 - V_4\}$

∴ $\boxed{\text{No of chromatic partitions} = 2}$